Notes Prepared

by

PREETHA VARGHESE

Syllabus for - S1 & S2 - Engineering Mathematics

MODULE 1

Matrix:

Elementary transformation - finding inverse and rank using elementary transformation - solution of linear equations using elementary transformation, eigen values and eigen vectors - Application of Cayley Hamilton theorem - Diagonalisation - Reduction of Quadratic form in to sum of squares using orthogonal transform.

MODULE - 2

Partial differentiation:

Partial differentiation - Chain rule - Euler's theorem for homogeneous function - Taylors series for functions of two variables - maxima and minima of function of two variables.

MODULE - 3

Multiple Integrals:

Double integration in Cartesian and polar co-ordinates - application in finding area and volume using double integrals - change of variable using Jacobian triple integrals in Cartesian, cylindrical and spherical co-ordinates - volume using triple integrals - simple problems.

MODULE - 4

Laplace Transforms

Laplace transforms - Laplace transform of derivatives and integrals - shifting theorem - differentiation and integration of transforms - inverse transforms - application of convolution property - solution of linear differential equations with constant coefficients using Laplace transform Laplace transform of unit step function, impulse function, and periodic function.

MODULE - 5

Fourier series

Dirichlet condition - Fourier series with period and 2L - HAlf range sine and cosine - series- simple problems -rms value.

Rank of a Matrix

If we select r rows and r columns from any matrix A, deleting all other rows and columns, then the determinent formed by these r x r elements is called the minor of A of order r. Clearly there will be number different minors of the same order, got by deleting different rows and columns from the same matrix.

The rank of a matrix is the largest order of any non-vanishing minor of the matrix.

i.e, A matrix is said to be of rank r when
(1) It has at least one non-zero minor of order r, and
(2) every minor of order higher than r vanishes.
The rank of the matrix A is denoted by ñ(A).

Elementary Transformation of a matrix:

The following operations are known as elementary transformations of a matrix.

1). The interchange of two rows (Columns)
[The interchanging of i^{th} and j^{th} rows (Columns), is denoted by Rij (Cij)

2). The multiplication of any row (Column) by a non zero number
[The multiplication of i^{th} row (Column) by k is denoted by kRi (Ci)

3). Addition of a constant multiple of the elements of any row (column) to the corresponding elements of any other row (column)
[Ri + pRj (Ci+pCj) denotes addition of i^{th} row (column) p-times the j^{th} row (column)

Normal form of a Matrix

Every non-zero matrix of rank r can be reduced by a sequence of elementary transformations to form

$$\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$$

called the normal form of A. Where I_r is the identity matrix of order 'r'.

EQUIVALENT MATRIX:

Two Matrices A and B are said to be equivalent if one can be obtained from, the other by a sequence of elementary transformations. Two equivalent matrices have the same order and the same rank. The symbol ~ is used for equivalence.

INVERSE FROM ELEMENTARY MATRICES:

The elementary row transformation which reduce a giver square matrix A to unit matrix, when applied to unit matrix I give the inverse of A.

For practical evaluation of A^{-1} the two matrices A and I are written side by side and the same row transformation are performed on both.

When A is reduced to I, the other matrix represents A^{-1}. This method is known as Gauss-Jordan method of finding the inverse.

Examples:

1). Find the rank of the following matrices.

(i) $\begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 2 \\ 2 & 6 & 5 \end{bmatrix}$ (ii) $\begin{bmatrix} 0 & 1 & -3 & -1 \\ 0 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}$

(iii) $\begin{bmatrix} 2 & 3 & 4 & -1 \\ 5 & 2 & 0 & -1 \\ -4 & 5 & 12 & -1 \end{bmatrix}$ (iv) $\begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 4 & 3 & 2 \\ 3 & 2 & 1 & 3 \\ 6 & 8 & 7 & 5 \end{bmatrix}$

(v) $\begin{bmatrix} 1 & -1 & 3 & 6 \\ 1 & 8 & -3 & -4 \\ 5 & 3 & 3 & 11 \end{bmatrix}$ (vi) $\begin{bmatrix} 2 & 3 & -1 & -1 \\ 1 & -1 & -2 & -4 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$

(vii) $\begin{bmatrix} 2 & -1 & 3 & 1 \\ 1 & 4 & -2 & 1 \\ 5 & 2 & 4 & 3 \end{bmatrix}$

Sol:

(i) Operating $R_2 \to R_2 - R1$ and $R_3 \to R_3 - 2R_1$ so that the given matrix

$$\sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & -1 \\ 0 & 2 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

Clearly the third order minor of A vanishes Also its second order minors formed by its second and third rows are all zero. The other second order minor is

$$\begin{bmatrix} 1 & 3 \\ 0 & -1 \end{bmatrix} = -1 \neq 0$$

$\therefore$ The rank of the given matrix = 2

(ii) Let A = $\begin{bmatrix} 0 & 1 & -3 & -1 \\ 0 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 0 & 0 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -3 & -1 \end{bmatrix}$

$C_3 \to C_3 - C_4$

$$\sim \begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

$\begin{bmatrix} R_3 \to R_3 - R_1 \\ R_4 \to R_4 - R_1 \end{bmatrix}$

$$\sim \begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\begin{bmatrix} R_3 \to R_3 - 3R_2 \\ R_4 \to R_4 - R_2 \end{bmatrix}$ $\begin{bmatrix} C_3 \to C_3 - 3C_2 \\ C_4 \to C_4 + C_2 \end{bmatrix}$

Clearly the second order minor of A is zero. Also every third order minor is zero. But of all the second order minors only.

$$\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = -1 \neq 0$$

$\therefore$ The rank of the given matrix is two

(iii)Let A= $\begin{bmatrix} 2 & 3 & 4 & -1 \\ 5 & 2 & 0 & -1 \\ -4 & 5 & 12 & -1 \end{bmatrix} \sim \begin{bmatrix} -1 & 3 & 4 & 2 \\ -1 & 2 & 0 & -1 \\ -4 & 5 & 12 & -1 \end{bmatrix} \sim \begin{bmatrix} -1 & 3 & 4 & 2 \\ 0 & -1 & -4 & 3 \\ 0 & 2 & 8 & 6 \end{bmatrix}$

$(C_3 \leftrightarrow C_4)$ $\begin{bmatrix} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R_1 \end{bmatrix}$

$$\sim \begin{bmatrix} -1 & 0 & -8 & 11 \\ 0 & -1 & -4 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

All the third order minors are zero but the second order minor

$\begin{vmatrix} -1 & 0 \\ 0 & -1 \end{vmatrix} = 1 \neq 0$ $\therefore$ Rank of the matrix = 2

(iv) Let A = $\begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 4 & 3 & 2 \\ 3 & 2 & 1 & 3 \\ 6 & 8 & 7 & 5 \end{bmatrix}$ $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 0 & -3 & 2 \\ 3 & -4 & -8 & 3 \\ 6 & -4 & -11 & 5 \end{bmatrix}$

$\begin{bmatrix} C_3 \to C_2 - 2C_1 \\ C_3 \to C_3 + 3C_1 \end{bmatrix}$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -3 & 2 \\ 0 & -4 & -8 & 3 \\ 0 & -4 & -11 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -3 & 2 \\ 0 & -4 & -8 & 3 \\ 0 & 0 & -3 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -3 & 2 \\ 0 & -4 & -8 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$(R_2 \to R_2 - 2R_1, R_3 \to R_3 - 3R_1, R_4 \to R_4 - 6R_1)$ $(R_4 \to R_4 - R_3)$ $(R_4 \to R_4 - R_3)$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & -4 & -2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & -2 & 7 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$(C_3 \to C_3+2C_4)$ $\quad$ $(C_1 \to C_1 -2C_3, C_4 \to C_4-7C_2)$

All the third order minoras except $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \neq 0$ $\therefore$ The rank of the given matrix =3

(v). Let A = $\begin{bmatrix} 1 & -1 & 3 & 6 \\ 1 & 8 & -3 & -4 \\ 5 & 3 & 3 & 11 \end{bmatrix}$ (H.W)

(vi) Let A= $\begin{bmatrix} 2 & -1 & 3 & 1 \\ 1 & 4 & -2 & 1 \\ 5 & 2 & 4 & 3 \end{bmatrix}$

$$\sim \begin{bmatrix} 1 & -1 & 1 & 0 \\ 5 & 4 & -3 & 5 \\ 7 & 2 & -1 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & 0 \\ 5 & 9 & -3 & 5 \\ 7 & 9 & -1 & 5 \end{bmatrix}$$

$(C_1 \to C_1 +C_2,\ C_3 \to C_3+C_1, C_4 \to C_2 +C_4)$ $\quad$ $(C_2 \to C_1 +C_2)$

$$\sim \begin{bmatrix} 1 & 0 & 1 & 0 \\ 5 & 1 & -3 & 1 \\ 7 & 1 & -1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & 0 \\ 5 & 1 & -3 & 0 \\ 7 & 1 & -1 & 0 \end{bmatrix}$$

$(C_2 \to \frac{C_2}{9},\ C_3 \to \frac{C_3}{5})$ $\quad$ $(C_4 \to C_4 -C_2)$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 5 & 1 & 8 & 0 \\ 7 & 1 & 8 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 5 & 1 & 1 & 0 \\ 7 & 1 & -1 & 0 \end{bmatrix}$$

$(C_3 \to C_1 -C_3)$ $\quad$ $(C_3 \to \frac{C_3}{8})$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 5 & 1 & 0 & 0 \\ 7 & 1 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 5 & 1 & 0 & 0 \\ 2 & 0 & 0 & 0 \end{bmatrix}$$

$(C_3 \to C_3 -C_2)$ $\quad$ $(R_3 \to R_3 -R_2)$

$\therefore$ Rank of the matrix is = 2

1. Using Gauss - Jordan method find the inverse of

$$\begin{bmatrix} 2 & 1 & 2 \\ 2 & 2 & 1 \\ 1 & 2 & 2 \end{bmatrix}$$

Sol :

$$\left[\begin{array}{ccc|ccc} 2 & 1 & 2 & 1 & 0 & 0 \\ 2 & 2 & 1 & 0 & 1 & 0 \\ 1 & 2 & 2 & 0 & 0 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & ½ & 1 & ½ & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & -3 & -2 & 1 & 0 & -2 \end{array}\right]$$

$(R_1 \rightarrow R_{½},\ R_2 \rightarrow R_{2} - R_1,\ R_3 \rightarrow R_1 \rightarrow 2R_3)$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 3/2 & 1 & -½ & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & -3 & -2 & 0 & 0 & 1 \end{array}\right]$$

$(R_1 \rightarrow R_{3/-5})$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 3/2 & 1 & -1/3 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & -5 & -2 & 3 & -2 \end{array}\right]$$

$(R_3 \rightarrow R_{3+3R_2})$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 3/2 & 1 & -½ & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 2/5 & -3/5 & 2/5 \end{array}\right]$$

$(R_3 \rightarrow R_{3/-5})$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2/5 & 2/5 & 3/5 \\ 0 & 1 & 0 & -3/5 & 2/5 & 2/5 \\ 0 & 0 & 1 & 2/5 & -3/5 & 2/5 \end{array}\right]$$

Hence the inverse of the given matrix is $\begin{bmatrix} 2/5 & 2/5 & -3/5 \\ -3/5 & 2/5 & 2/5 \\ 2/5 & -3/5 & 2/5 \end{bmatrix}$

2. Using Gauss - Jordan method find the inverse of

$$\begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix}$$

Sol:

$$\left[\begin{array}{ccc|ccc} 0 & 1 & 2 & 1 & 0 & 0 \\ 1 & 2 & 3 & 0 & 1 & 0 \\ 3 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 3 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & -5 & -8 & 0 & -3 & 1 \end{array}\right]$$

$(R_1 \leftrightarrows R_2)$ $(R_3 \rightarrow R_{3} - 3R_1)$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & -2 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 2 & 5 & -3 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & -2 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 5/2 & -3/2 & ½ \end{array}\right]$$

$(R_1 \rightarrow R_1 - 3R_2, R_3 \rightarrow R_3+5R_2)$ $(R_3 \rightarrow R_3/_2)$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & ½ & -½ & ½ \\ 0 & 1 & 0 & -4 & 3 & -1 \\ 0 & 0 & 1 & 5/2 & -3/2 & ½ \end{array}\right]$$

Hence the inverse of the given matrix is

$$\begin{bmatrix} ½ & -½ & ½ \\ -4 & 3 & -1 \\ 5/2 & -3/2 & ½ \end{bmatrix}$$

1. Using Gauss - Jordan method find the inverse of the following matrix

$$A= \begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -2 & -4 \end{bmatrix}$$

Solution

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 3 & 1 & 0 & 0 \\ 1 & 3 & -3 & 0 & 1 & 0 \\ -2 & -4 & -4 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 1 & 3 & 1 & 0 & 0 \\ 0 & 2 & -6 & -1 & 1 & 0 \\ 0 & -2 & 2 & 2 & 0 & 1 \end{array}\right]$$

$(R_2 \rightarrow R_2 - R_1 \; R_3 \rightarrow R_3+2R_1)$ $(R_2 \rightarrow ½ \; R_2, R_3 \rightarrow ½R_3)$

$$\sim \left[\begin{array}{ccc|ccc} 0 & 1 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -½ & ½ & 0 \\ 0 & -1 & 1 & 1 & 0 & ½ \end{array}\right]$$

$(R_1 \rightarrow R_1 - R_2, R_3 \rightarrow R_3+R_2)$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 6 & 3/2 & -½ & 0 \\ 0 & 1 & -3 & -½ & ½ & 0 \\ 0 & 0 & -2 & ½ & ½ & ½ \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & 1 & 3/2 \\ 0 & 1 & 0 & -5/4 & -1/4 & -3/4 \\ 0 & 0 & 1 & -1/4 & -1/4 & -1/4 \end{array}\right]$$

$(R_1 \rightarrow R_1+3R_3, R_2 \rightarrow R_2 - 3/2R_3, R_2 \rightarrow -½R_2)$

Hence

$$A^{-1} = \begin{bmatrix} 3 & 1 & 3/2 \\ -5/4 & -1/3 & -3/4 \\ -1/4 & -1/4 & -1/4 \end{bmatrix}$$

CONSISTENCY OF A SYSTEM OF LINEAR EQUATIONS:

Consider a system of m linear equations

$$\left.\begin{array}{l} a_{11}x_1+a_{12}x_2 + \text{----------------} + a_{1n}\, x_n = k_1 \\ a_{21}x_1+a_{22}x_2 + \text{----------------} + a_{2n}\, x_n = k_1 \\ \text{--} \\ \text{--} \\ a_{m1}x_1+a_{m2}x_2 + \text{----------------} + a_{mn}\, x_n = k_1 \end{array}\right| \text{------------ (1)}$$

Containing n unknown x_1, x_2 ------------, x_n

To determine whether the equation (1) are consistent (ie, possess solution) or not , are consider the ranks of the matrices,

$$A = \begin{bmatrix} a_{11} & a_{12} & - & - & a_{13} \\ a_{21} & a_{22} & - & - & a_{22} \\ - & - & - & - & - \\ - & - & - & - & - \\ a_{m1} & a_{m1} & - & - & a_{mn} \end{bmatrix} \text{ and } K = \begin{bmatrix} a_{11} & a_{12} & --- & a_{1n} & k_1 \\ a_{21} & a_{22} & --- & a_{2n} & k_2 \\ - & - & --- & - & - \\ - & - & --- & - & - \\ a_{m1} & a_{m1} & - & a_{mn} & k_m \end{bmatrix}$$

A is the co efficient matrix and k is called the augmented matrix of the equations (1)

The system of equations (1) consistent if and only if the co efficient matrix A and the augumented matrix k are of the same rank, otherwise the system is inconscistent.

SYSTEM OF LINEAR HOMOGENIOUS EQUATIONS

Consider homogenious linear equations

$$\left.\begin{array}{l} a_{11}x_1+a_{12}x_2 + \text{----------------} + a_{1n}\, x_n = 0 \\ a_{21}x_1+a_{22}x_2 + \text{----------------} + a_{2n}\, x_n = 0 \\ \text{--} \\ \text{--} \\ a_{n1}x_1+a_{n2}x_2 + \text{----------------} + a_{nn}\, x_n = 0 \end{array}\right| \text{------------ (1)}$$

Hence we can write the co-efficient matrix A and reduce it to the triangular form by eliminating row operations.

1. If ñ = n, the equations (1) have only a trivial solution

   $x_1 = x_2 =$ .... $x_n = 0$

   If ñ < n, the equations have (n-ñ) independent solutions and r can not be > n.

   The number of linearly independent solutions is (n-ñ) means, if arbitary values areas signed to (n-ñ) of the variables, the values of the remaining variables can be uniquely found.

2. When m<n, (ie, the number of equations is less than the number of variables), the solution is always other than $x_1 = x_2 =$ .... $x_n = 0$

3. When m = n (ie, the number of equations = the number of variables), the necessary condition for solutions other than $x_1 = x_2 = \ldots\ldots\ldots\ldots\ldots\ldots\ldots x_n = 0$ is that the determinant of the co- efficient matrix is zero. In this case the equations are said to be consistent and such a solution is called non-trivial solution. The determination is called the eliminant of the equations.

CHARACTERISTIC EQUATION

If A is any square matrix of order n, we can form the matrix A = = ë*I*, where I is the nth order unit matrix. The determinent of this matrix ie $|A - \ddot{e}I| = 0$ is called the characteristic equation of A.

If $A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$

Then

$$A - \ddot{e}I = \begin{vmatrix} a_1 - \ddot{e} & b1 & c_1 \\ a_2 & b_2 - \ddot{e} & c_2 \\ a_3 & b_3 & c_3 - \ddot{e} \end{vmatrix}$$

Then the characteristic equation of A is

$$\begin{vmatrix} a_1 - \ddot{e} & b1 & c_1 \\ a_2 & b_2 - \ddot{e} & c_2 \\ a_3 & b_3 & c_3 - \ddot{e} \end{vmatrix} = 0$$

The roots of this equation are called characteristic roots or eigen values of A

EIGEN VECTORS

Corresponding to each root of $|A - \ddot{e}I| = 0$ the homogenious system of n linear equations,

$(a_{11} - \ddot{e})\, x_1 + a_{12}x_2 + \text{---------------} + a_{1n}x_n = 0$

$a_{21}x_1 + (a_{22} - \ddot{e})\, x_2 + \text{----------------} + a_{2n}x_n = 0$

$a_{n1}x_1 + a_{n2}x_2 + \text{-------------------} + a_{nn} - \ddot{e})\, x_n = 0$

Has a non - zero solution

$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ which is called the eigen vector

Properties of eigen values:

1. The sum of the eigen values of a matrix is the sum of the elements of the principal diagonal.
2. If ë is an eign value of a matrix A, then (1/ ë) is the eign value of A^{-1}
3. If ë is an eign value of an orthogonal matrix then (1/ ë) is also itseigen value.
4. If $ë_{1,}$ $ë_2$ ....$ë_n$ are the eigen values of a matrix A, then A^m has the eigen values $ë_1^m$, $ë_2^m$, --------------- $ë_n^m$, (m being a positive integr).

Cayley Hamilton Theorem:

Every square matrix satisfies its own charecteristic equation.

Ie, if the charecteristic equation for the n^{th} order square matrix A is

$|A-ëI| = (-1)^n ë^n + k_1 ë^{n-1} + ... + k_n = 0$

Then $(-1)^n A^n + k_1 A^{n-1} + ... + k_n = 0$

Cor:

Multiplying above equation by A^{-1}, and simplyfying we get

$$A^{-1} = -\left[\frac{(-1)^n A^{n-1} + k_1 A^{n-2} + ... + k_n A^{-1}}{k_n}\right]$$

Examples:

1. Find the eigen values of and eigen vectors of the matrices

1. $\begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix}$

2. $\begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$

3. $\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$

4. $\begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$

2. The ch. equation is given by |A-ë*I*| = 0

ie,

-2-ë	2	-3
2	1-ë	-6
-1	-2	0-ë

= 0

ë3+ë2-21ë - 45 = 0

ë = -3, -3, 5

Thus the eagen values are -3, -3, 5

Corresponding to

ë = -3

the eigenvectors are given by (A+3I)X = 0

Or

$$\begin{bmatrix} 1 & 2 & -3 \\ 2 & 4 & -6 \\ -1 & -2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

We get only one independent equation $x_1+2x_2-3x_3 = 0$

Choosing $x_2 = 0$ we get $x_1-3x_3 = 0$

$$\frac{x_1}{3} = \frac{x_2}{0} = \frac{x_3}{1}$$

giving eigenvectors (3, 0, 1)

Choosing $x_3 = 0$ we get $x_1+2x_2 = 0$

$$\frac{x_1}{2} = \frac{x_2}{-1} = \frac{x_3}{0}$$

giving eigenvectors (2, -1,0)

Any other eigenvector corresponding to ë = -3 will be the linear.

Corresponding to ë = 5 the eigenvectors are given by

$$\begin{bmatrix} -7 & 2 & -3 \\ 2 & -4 & -6 \\ -1 & -2 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

ie,

$$7x_1+2x_2-3x_3 = 0$$
$$2x_1-4x_2-6x_3 = 0$$
$$-x_1-2x_2-5x_3=0$$

$$\frac{x_1}{-12-12} \quad \frac{x_2}{-6-42} \quad \frac{x_3}{28-4}$$

$$\frac{x_1}{1} = \frac{x_2}{2} = \frac{x_3}{-1}$$

giving the eigen vector (1, 2, -1)

3. Let A = $\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$

The characteristic equation of A is

$|A - \ddot{e}I| = 0$

$$\begin{vmatrix} 8-\ddot{e} & -6 & 2 \\ -6 & 7-\ddot{e} & -4 \\ 2 & -4 & 3-\ddot{e} \end{vmatrix} = 0$$

$\ddot{e}^{3} - 18\ddot{e}^{2} + 45\ddot{e} = 0$

ë = 0, 3, 15 are the eigenvalues

The Eigenvector corresponding to ë = 0 is given by (A-01)X=0

$$\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$8x_1 - 6x_2 + 2x_3 = 0$
$-6x_1 + 7x_2 - 4x_3 = 0$
$2x_1 - 4x_2 + 3x_3 = 0$

The coefficient matrix of these equations is of rank 2

∴ The equation possess only one linearly independent solution

$$\frac{x_1}{24-14} \quad \frac{x_2}{32-12} \quad \frac{x_3}{56-36}$$

$$\frac{x_1}{1} = \frac{x_2}{2} = \frac{x_3}{2}$$

giving the eigenvector (1, 2, 2)

Corresponding to ë = 3 the eigen values are given by (A-3I)X = 0

$$\begin{bmatrix} 5 & -6 & 2 \\ -6 & 4 & -4 \\ 2 & -4 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$5x_1 - 6x_2 + 2x_3 = 0$
$-6x_1 + 4x_2 - 4x_3 = 0$
$2x_1 - 4x_2 = 0$

Here also these equations possess only one linear independent solution

So the eign vector corresponding to ë = 3 is (2, 1, -2)

Similarly the eigenvector corresponding to ë = 15 is (2, -2, 1)

4. Let A = $\begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$

Then its characteristic equation is given by $|A - \ddot{e}I| = 0$

ie, $$\begin{vmatrix} 6-\ddot{e} & -2 & 2 \\ -2 & 3-\ddot{e} & -1 \\ 2 & -1 & 3-\ddot{e} \end{vmatrix} = 0$$

i.e, ë^{3}-12ë^{2}+ 36ë -32=0

i.e, ë = 2,2,8.

Let ë = 2. Then the eagen vectors corresponding to it is given by the equations.

$4x_1-2x_2+2x_3 = 0$

$-2x_1+x_2-x_3 = 0$

$2x_1-x_2+x_3 = 0$

Thus there we get only one equation $2x_1-x_2+x_3 = 0$

No take $x_2 = 0$

Then $2x_1-x_3 = 0$

$$\therefore \quad \frac{x_1}{1} = \frac{x_2}{0} = \frac{x_3}{-2}$$

i.e, (1, 0, -2) is an egen vector corresponding to ë = 2

Now take $x_3 = 0$

Then $2x_1 = x_2$

$$\therefore \quad \frac{x_1}{1} = \frac{x_2}{2} = \frac{x_3}{0}$$

ie, (1, 2, 0) is an egen vector corresponding to ë = 2

ë = 8 Then the corresponding eigen vector is given by the equations.

$-2x_1-2x_2+2x_3 = 0$

$-2x_1-5x_2-x_3 = 0$

$2x_1-x_2-5x_3 = 0$

ie, y = -z

$x_1+x_2-x_3 = 0$

$$\therefore \quad \frac{x_1}{2} = \frac{x_2}{-1} = \frac{x_3}{1}$$

(2,-1,1) is the corresponding eigen vector.

6.Verify cayley Hamilton theorem for the matrix $A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$ and hence compute A^{-1}

Solution:

The characteristic equation of A is |A - ëI| = 0

$$\begin{bmatrix} 2-ë & -1 & 1 \\ -1 & 2-ë & -1 \\ 1 & -1 & 2-ë \end{bmatrix} = 0$$

or ë^{1} -6ë^{2}+9ë -4=0

To verify Cayley Hamilton Theorem we have to show that

$A^3 -6A^2+9A-4= 0$

$$A^2 = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix}$$

$$A^3 = A^2A \quad \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix} \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 22 & -22 & -21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{bmatrix}$$

$$\therefore \quad A^3 - 6A^2 + 9A - 4I = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

This verifies the Cayley Hamilton theorem.

Now from $\therefore \quad A^3 - 6A^2 + 9A - 4I = 0$

Multiplying both sides by A^{-1} we get

$A^2 - 6A + 9 - A^{-1} = 0$

$\rightarrow$ 4A-1=A2 -6A+9*l*

$$4A^{-1} = \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix} - 6 \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} + 9 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\therefore \quad A^{-1} = \frac{1}{4} \begin{bmatrix} 3 & -1 & -1 \\ -1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}$$

Diagonalisation

Result - 1

If a square matrix A of order has n linearly independent eigen vectors, then a matrix P can be found such that $P^{-1}AP = D$, a diagonal matrix.

Modal matrix and spectral matrix:

The matrix P which P which diagonalises A is called the modal matrix and the resulting diagonal matrix D is known as spectral matrix.

Remark:

The modal matrix of A i.e, P is found by grouping the eigenvectors of A into a square matrix.

ie, If $X_1 = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}$, $X_2 = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}$ and $X_3 = \begin{bmatrix} x_3 \\ y_3 \\ z_3 \end{bmatrix}$

are the eigenvectors of a square matrix A of order 3 then $P = \begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{bmatrix}$

Remark:

$$P^{-1}AP \quad \begin{bmatrix} ë_1 & 0 & 0 \\ 0 & ë_2 & 0 \\ 0 & 0 & ë_3 \end{bmatrix}$$, Where $ë_1, ë_2, ë_3$ are the eigen values of A.

Similar matrics:

A square matrix A is said to be similar to a square matrix B of order ‘n’ if there exists matrix P of order ‘n’ such that B = P^{-1} AP.

Result

Similar matrices have same eigen values.

Use of diagonalisation for finding the powers of a square matrix.

Let A be the any square matrix and let P and D be the corresponding modal matrix and spectral matrix.

Then $D = P^{-1}AP$

Now, $D^n = (P^{-1}AP)\,(P^{-1}AP)(P^{-1}AP)\,......(P^{-1}AP)$(n-times)

$= (P^{-1}A^nP.$ Thus, $A^n = PD^nP^{-1}.$

Remark

If $D = \begin{bmatrix} \ddot{e}_1 & 0 & 0 \\ 0 & \ddot{e}_2 & 0 \\ 0 & 0 & \ddot{e}_3 \end{bmatrix}$ Then $D^n \begin{bmatrix} \ddot{e}_1{}^n & 0 & 0 \\ 0 & \ddot{e}_2{}^n & 0 \\ 0 & 0 & \ddot{e}_3{}^n \end{bmatrix}$

Quadratic forms:

A homogeneous expression of the second degree in any number of variables is called a quadratic form.

Examples

1. $ax^2+2hxy+by^2$
2. $ax^2+by^2+cz^2+2hxy+2gyz+2fzx.$

Remark:

Note that above quadratic expression can be represented in the matrix form by

1. $ax^2+2hxy+by^2 = \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} a & h \\ h & b \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$

2. $ax^2+by^2+cz^2+2hxy+2gyz+2fzx = \begin{bmatrix} x & y & z \end{bmatrix} \begin{bmatrix} a & h & f \\ h & b & g \\ f & g & c \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

Canonical forms:

If a real quadratic form be expressed in a sum of squares of new variables by means of any non- singular linear transformation then the later quadratic expression is called a canonical form of the given quadratic expression.

Reduction of Quadratic form to canonical form:

Let $X^!AX$ be any Quadratic expression where A is a square matrix of order 'n'.
Let P be the modal matrix and D be the spectral matrix of A.

$$D = P^{-1}AP$$

$$X^!(P^{-1}AP)X = X^!DX = \sum_{i=i}^{n} \ddot{e}_i x_i^2$$

Where $\ddot{e}_1, \ddot{e}_2, \ddot{e}_3$ are the eigen values of A and X = $X = \begin{bmatrix} x_1 \\ x_2 \\ - \\ - \\ x_n \end{bmatrix}$

Thus the Quadratic form reduces to a sum of squares $\sum_{i=i}^{n} \ddot{e}_i x_i^2$ and P is the matrix of transformation.

Which is an orthogonal matrix. Thus the above reduction is called orthogonal transformation.

Examples:

1. Diagonalise the following matrices.

1. $\begin{bmatrix} -1 & 2 & -2 \\ 1 & 2 & 1 \\ -1 & -1 & 0 \end{bmatrix}$

2. $\begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix}$

Sol:

1. The characteristic equation is

$$A - ëI = \begin{vmatrix} -1-ë & 2 & -2 \\ 1 & 2-ë & 1 \\ -1 & -1 & -ë \end{vmatrix} = 0$$

$ë^3 - ë^2 - 5ë + 6 = 0$

Solving it we have $ë = 1, \pm\sqrt{5}$

When $ë = 1$, the corresponding eigen vector is given by

-2x+2y-2z = 0

x-y-z = 0

x-y+z = 0

$$\therefore \quad \frac{x}{2} = \frac{y}{0} = \frac{z}{-2}$$

giving the eigen vector (1, 0, -2)

When $ë = \sqrt{5}$ the corresponding eigenvector is given by

$(-1 - \sqrt{5})x + 2y - 2z = 0$

x + y = z = 0

x - y + z = 0

$$\therefore \quad \frac{x}{6-2\sqrt{5}} \quad \frac{y}{-1+\sqrt{5}} \quad \frac{z}{1-\sqrt{5}}$$

i.e, $\dfrac{x}{\sqrt{5}-1} = \dfrac{y}{1} = \dfrac{z}{-1}$

giving the eigen vector $(\sqrt{5} - 1, 1, 1)$

Similarly When $= ë - \sqrt{5}$ the corresponding eigen vector is $(\sqrt{5} + 1, 1, -1)$

Thus the modal matrix is givenn by

$$P = \begin{bmatrix} 1 & \sqrt{5}-1 & \sqrt{5}+1 \\ 0 & 1 & 1 \\ -1 & -1 & 1 \end{bmatrix}$$

And the spectral matrix

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sqrt{5} & 0 \\ 0 & 0 & -\sqrt{5} \end{bmatrix}$$

Reduse the quadratic form $3x^2+3y^2+3z^2-2yz+2zx+2xy$ to the canonicalform. Also specify the metrix of transformation.

The matrix of the givenqudratic form is $A = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 3 & -1 \\ 1 & -1 & 3 \end{bmatrix}$

Now we have to find the eigen values of A. For this consider the characteristic equation of A. Then its characteristic equation is given by $|A - ëI| = 0$

i.e, $\begin{vmatrix} 3-ë & 1 & 1 \\ 1 & 3-ë & -1 \\ 1 & -1 & 3-ë \end{vmatrix} = 0$

i.e, $ë^3-9ë^2+24ë-16 = 0$ i.e, $ë = 1, 4, 4.$

Hence the given quadratic form reduces to cononical form $x^2+4y^2+4z^2$

Let $ë = 1$. Then the eigen vectors corresponding to it is given by the equations.

$2x+y+z = 0$

$x+2y-z = 0$

$x - y +2z = 0$

i.e, $\frac{x}{1} \quad \frac{y}{-1} \quad \frac{z}{-1}$

i.e, the eigen vector is (-1, 1, 1) and its normalized form is $\left| \frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right|$

$ë = 4$, $x - y -z = 0$

If $x = 0$

The corresponding characteristic vector is (0, 1, -1) and its normalized form is $0, \frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}$

If $y = z$, The corresponding characteristic vector is (2,1,1) and its normalized form is $\frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}$

Hence the matrix of transformations $P = \begin{bmatrix} \frac{-1}{\sqrt{3}} & 0 & \frac{2}{\sqrt{6}} \\ \frac{-1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \end{bmatrix}$

3. Reduse the quadratic form $3x^2+5y^2+3z^2-2yz+2zx-2xy$ to the canonicalform. Also specify the metrix of transformation.

The matrix of the givenqudratic form is $A = \begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$

Now we have to find the eigen values of A. For this consider the characteristic equation of A. Then its characteristic equation is given by $|A - ëI| = 0$

i.e, $\begin{vmatrix} 3-ë & -1 & 1 \\ -1 & 5-ë & -1 \\ 1 & -1 & 3-ë \end{vmatrix} = 0$

ie., ë = 2, 3, 6

Hence the given quadratic formreduces to cononicl form $2x^2+3y^2+6z^2$

Let ë = 2 . Then the eagen vector corresponding to it is given by the equations.

$x-y+z = 0$

$-x+3y-z = 0$

$-x +3y -z = 0$

$x - y +z = 0$

ie, $\frac{x}{1} = \frac{y}{0} = \frac{z}{-1}$

ie, the eigen vector is (1, 0, -1) and its normalized form is $\left| \frac{1}{\sqrt{2}}, 0, \frac{-1}{\sqrt{2}} \right|$

Let ë = 3, The corresponding eigen vector is (1,1,1) and its normalized form is $\left| 0, \frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}} \right|$

If y = z, The corresponding eigen vector is (2, 1, 1) and its normalized form is $\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}$

ë = 6 The corresponding eigen vector is (1, -2, 1) and its normalized form is $\frac{1}{\sqrt{6}}, \frac{-2}{\sqrt{6}}, \frac{1}{\sqrt{6}}$

Hence the matrix of transformation is $P = \begin{bmatrix} \frac{1}{\sqrt{2}}, & \frac{1}{\sqrt{3}}, & \frac{1}{\sqrt{6}} \\ 0, & \frac{1}{\sqrt{3}}, & \frac{-2}{\sqrt{6}} \\ -\frac{1}{\sqrt{2}}, & \frac{1}{\sqrt{3}}, & \frac{1}{\sqrt{6}} \end{bmatrix}$