**MODULE - II**

**Partial Differentiation**

**(FUNCTIONS OF TWO OR MORE VARIABLES)**

## 3.1. FUNCTIONS OF TWO VARIABLES

If three variables x, y, z are so related that the value of z depends upon the values of x and y, then z is called a function of two variables x and y, and this is denoted by z = $f$(x, y).

z is called the dependent variable while x and y are called independent variables.

For example, the area of a triangle is determined wheb its base and altitude are known. Thus area of triangle is a function of two variables, base and altitude.

(In a similar way, a function of more than two variables can be defined).

**Geometrically**. Let z = $f$(x, y) be a function of two independent variables x and y defined for all pairs of values of x and y which belong to an area A of the xy - plane. Then to each point (x, y) of this area corresponds a value of z given by the relation z = $f$(x, y). Representing all these values (x, y, z) by points in space, we get a surface.

**Hence the function z *f*(x, y) represents a surface.**

## 3.2. PARTIAL DERIVATIVES OF FIRST ORDER

Let z = $f$(x, y) be a function of two independent variables x and y. If y is kept constant and x alaone is allowed to vary, then z becomes a function of x only. The derivative of z,with respect to x, treating y as constant, is called partial derivative of z w.r.t.x and is denoted by

$$\frac{\partial z}{\partial x} \quad \text{or} \quad \frac{\partial f}{\partial x} \quad \text{or} \quad f_x$$

$$\text{Thus} \quad \frac{\partial z}{\partial x} = \underset{h \to 0}{\text{Lt}} \frac{f(x+h, y) - f(x, y)}{h}$$

Similarly the derivative of z, with respect to y, treating x as constant, is called partial derivative of z w.r.t. y and is denoted by

$$\frac{\partial z}{\partial y} \quad \text{or} \quad \frac{\partial f}{\partial y} \quad \text{or} \quad f_y$$

$$\text{Thus} \quad \frac{\partial z}{\partial y} = \underset{k \to 0}{\text{Lt}} \frac{f(x, y+k) - f(x, y)}{k}$$

$\frac{\partial z}{\partial x}$ $\frac{\partial z}{\partial y}$ are called first order partial derivatives of z.

[In general, if z is a function of two or more independant variables, then the partial derivative of z w.r.t. any one of the independent variables is the ordinary derivative of z w.r.t that variable, treating all other variables as constant].

[Geometrically, Let z = f(x, y) be a function of two variables x and y. Then by Art. 3, 1 it represents a surface S. If y = k, a constant, thewn y = k represents a plane parallel to the zx - plane.

$\therefore \quad z = f(x, y)$ and $y = k$

represent a plane curve C which is the section of S by y = k.

$\frac{\partial z}{\partial x}$ represents the slope of tangent to C at (x, k, z)

Thus $\frac{\partial z}{\partial x}$ gives the slope of the tangent drawn to the curve of intersection of the surface $z = f(x, y)$ and a plane parallel to zx - plane.

Similarly $\frac{\partial z}{\partial y}$ gives the slope of the tangent drawn to the curve of intersection of the surface $z = f(x, y)$ and a a plane parallel to yz - plane.

## 3.3. PARTIAL DERIVATIVES OF HIGHER ORDER

Since the first order partial derivatives $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ are themselves functions of x and y, they can be further differentiated partially w.r.t.x as well as y. These are called second order partial derivatives of z. The usual notations for these second order partial derivatives are :

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial x^2} \text{ or } f_{xx}$$

$$\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial y^2} \text{ or } f_{yy}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial x \partial y} \text{ or } f_{xy}$$

$$\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial y \partial x} \text{ or } f_{yx}$$

In general, $\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x}$ or $f_{xy} = f_{yx}$

Note 1. If $z = f(x)$, a function of single independent variable x, we get $\frac{dz}{dx}$

If $z = f(x, y)$, a function of two independent variables x and y, we get $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$

Similarly, for a function of more than two independent variables $x_1, x_2, \ldots\ldots, x_n$ we get

$$\frac{\partial z}{\partial x_1}, \frac{\partial z}{\partial x_2}, \ldots\ldots, \frac{\partial z}{\partial x_n}$$

**Note 2.** (i) If z = u +v, where u = $f$(x, y), v = $\phi$ (x, y), then z is a function of x and y.

$$\therefore \quad \frac{\partial z}{\partial x} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial x} \; ; \quad \frac{\partial z}{\partial y} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

(ii) If z = uv, where u = $f$(x, y), v = $\phi$ (x, y),

then

$$\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}(uv) = u\frac{\partial v}{\partial x} + v\frac{\partial u}{\partial x}$$

$$\frac{\partial z}{\partial y} = \frac{\partial}{\partial y}(uv) = u\frac{\partial v}{\partial y} + v\frac{\partial u}{\partial y}$$

(iii) If $z = \frac{u}{v}$, where u = $f$(x, y), v = $\phi$ (x, y),

then

$$\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}\left(\frac{u}{v}\right) = \frac{u\frac{\partial u}{\partial x} - u\frac{\partial v}{\partial x}}{v^2}$$

$$\frac{\partial z}{\partial y} = \frac{\partial}{\partial y}\left(\frac{u}{v},\right) = \frac{v\frac{\partial u}{\partial y} - u\frac{\partial v}{\partial y}}{v^2}$$

(iv) If z = $f$ (u), where u = $\phi$ (x, y),

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}, \frac{\partial u}{\partial x} \; ; \quad \frac{\partial z}{\partial y} = \frac{\partial u}{\partial u}, \frac{\partial u}{\partial y}$$

## ILLUSTRATIVE EXAMPLES

**Example 1.** Find the first order partial derivatives of the following:

(i) $u = y^x$ (ii) $u = \tan^{-1} \frac{x^2+y^2}{x + y}$

(iii) $u = \cos^{-1}\left(\frac{x}{y}\right).$

**Sol .** (i) $u = y^x$

[Treating y as constant, u is of the form $a^x$]

$$\frac{\partial u}{\partial x} = y^x \log y$$

[Treating x as constant, u is of the form $y^n$]

$$\frac{\partial u}{\partial y} = xy^{x-1}$$

(ii) $u = \tan^{-1} \frac{x^2+y^2}{x+y}$

$$\frac{\partial u}{\partial x} = \frac{1}{1+\left(\frac{x^2+y^2}{x+y}\right)^2} , \frac{\partial}{\partial x}\left(\frac{x^2+y^2}{x+y}\right)$$

$$= \frac{(x+y)^2}{(x+y)^2+(x^2+y^2)^2} \; \frac{(x+y)\frac{\partial}{\partial x}(x^2+y^2) - (x^2+y^2)\frac{\partial}{\partial x}(x+y)}{(x+y)^2}$$

$$= \frac{(x+y).\,2x - (x^2+y^2).\,1}{(x+y)^2+(x^2+y^2)^2}$$

$$\Rightarrow \quad \frac{\partial u}{\partial x} = \frac{x^2+2xy-y^2}{(x+y)^2+(x^2+y^2)^2}$$

[Since u remains the same if we interchange x and y, u is symmetrical w. r.t.x and y. Interchanging x and y in (1), we have]

Similarly, $\frac{\partial u}{\partial y} = \frac{y^2+2xy-x^2}{(x+y)^2+(x^2+y^2)^2}$

(iii) $u = \cos^{-1}\left(\frac{x}{y}\right)$

$$\frac{\partial u}{\partial x} = \frac{-1}{\sqrt{1-\left(\frac{x}{y}\right)^2}} \frac{\partial}{\partial x}\left(\frac{x}{y}\right)$$

$$= \frac{-y}{\sqrt{y^2-x^2}} \quad \frac{1}{y} \quad \frac{-1}{\sqrt{y^2-x^2}}$$

$$\frac{\partial u}{\partial y} = \frac{-1}{\sqrt{1-\left(\frac{x}{y}\right)^2}} \frac{\partial}{\partial y}\left(\frac{x}{y}\right)$$

$$= \frac{-y}{\sqrt{y^2-x^2}}\left(\frac{-x}{y^2}\right) = \frac{x}{y\sqrt{y^2-x^2}}$$

**Example 2.** If $z(x+y) = x^2+y^2$, show that

$$\left[\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right]^2 = 4\left[1 - \frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right]$$

Sol. $z = \frac{x^2+y^2}{x+y}$ (z is symmetrical w.r.t.x and y)

$$\frac{\partial z}{\partial x} = \frac{(x+y)\frac{\partial}{\partial x}(x^2+y^2)\ x^2+y^2) - (x^2+y^2)\frac{\partial}{\partial x}(x+y)}{(x+y)^2}$$

$$= \frac{(x+y)\,.\,2x - (x^2+y^2)\,.1}{(x+y)^2} = \frac{x^2+2xy-y^2}{(x+y)^2}$$

Similarly, $\frac{\partial z}{\partial y} = \frac{y^2+2xy-x^2}{(x+y)^2}$

Now $\left[\frac{\partial z}{\partial x} = \frac{\partial z}{\partial y}\right]^2 = \left[\frac{2x^2-2y^2}{(x+y)^2}\right] = \frac{4(x+y)^2(x-y)^2}{(x+y)^4}$

$$= \frac{4(x-y)^2}{(x+y)^2}$$

$$4(1-\frac{\partial z}{\partial x} = \frac{\partial z}{\partial y} = 4\left[\frac{x^2+2xy-y^2)}{(x+y)^2} - \frac{y^2+2xy-x^2}{(x+y)^2}\right]$$

$$= 4\left[1 - \frac{x^2+2xy+y^2 - x^2-2xy+y^2-y^2-2xy+x^2}{(x+y)^2}\right]$$

$$= \frac{4(x^2-2xy+y^2)}{(x+y)^2}\ \frac{4(x-y)^2}{(x+y)^2}$$

$$\therefore \left[\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right]^2 = 4\left[1 - \frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right]$$

**Example 3.** If u = log (tan x + tan y), prove that

$$\text{Sin } 2x \frac{\partial u}{\partial x} + \text{Sin } 2y \frac{\partial u}{\partial y} = 2$$

Sol. u = log (tan x + tan y) (u is symmetrical w.r.t.x and y)

$$\frac{\partial u}{\partial x} = \frac{1}{\tan x + \tan y.} \frac{\partial}{\partial x}(\tan x + \tan y) = \frac{\sec^2 x}{\tan x + \tan y}$$

Similarly $\frac{\partial u}{\partial x} = \frac{\sec^2 y}{\tan x + \tan y}$

$$\text{Sin } 2x \frac{\partial u}{\partial x} + \sin 2y \frac{\partial u}{\partial y}$$

$$= \frac{1}{\tan x + \tan y.} (\sin 2x \sec^2 x + \sin 2y \sec^2 y)$$

$$= \left[\frac{1}{\tan x + \tan y.} \left(2 \sin x \cos x \frac{1}{\cos^2 x} + 2 \sin y \cos y. \frac{1}{\cos^2 y}\right)\right]$$

$$= \frac{2(\tan x + \tan y)}{\tan x + \tan y} = 2.$$

**Example 4.** If $f(x, y, z) = \begin{vmatrix} x^2 & y^2 & z^2 \\ x & y & z \\ 1 & 1 & 1 \end{vmatrix}$, prove that

$fx + fy + fz = 0.$

Sol. $f(x, y, z) = \begin{vmatrix} x^2 & y^2 & z^2 \\ x & y & z \\ 1 & 1 & 1 \end{vmatrix}$

Expanding by first row = $x^2 (y-z) - y^2 (x-z) + z^2 (x-y)$

$= x^2y - xy^2 + y^2z - yz^2 + z^2x - zx^2$

$fx = 2xy - y^2 + z^2 - 2zx$

$fy = x^2 - 2xy + 2yz - z^2$

$fz = y^2 - 2yz + 2zx - z^2$

$\therefore \quad fx + fy + fz = 0$

**Example 5.** Verify that $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$ for the following functions:

(i) $u = e^{ax} \sin by$ (ii) $u = \log \left| \frac{(x+y)^2}{xy} \right|$

(iii) $u = \tan^{-1} \left| \frac{x}{y} \right|$

**Sol. (i)** $u = e^{ax} \sin by$

$$\frac{\partial u}{\partial x} = \sin by.\ e^{ax} a = ae^{ax} \sin by$$

$$\frac{\partial u}{\partial y} = e^{ax} \cos by.\ b = be^{ax} \cos by$$

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial x} \left| \frac{\partial u}{\partial y} \right| = b \cos by.\quad e^{ax} . \partial = abe^{ax} \cos by$$

$$\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial}{\partial y} \left| \frac{\partial u}{\partial x} \right| = ae^{ax} \cos by.\ b = abe^{ax} \cos by.$$

$$\therefore \quad \frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$$

(ii) $u = \log \left| \frac{x^2 + y^2}{xy} \right| = \log (x^2 + y^2) - \log x - \log y$

$$\frac{\partial u}{\partial x} = \frac{2x}{x^2 + y^2} - \frac{1}{x}$$

$$\frac{\partial u}{\partial y} = \frac{2y}{x^2 + y^2} - \frac{1}{y}$$

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial x} \left| \frac{\partial u}{\partial y} \right| = \frac{\partial}{\partial x} \left[ 2y (x^2 + y^2)^{-1} \frac{-1}{y} \right]$$

$$= -2y (x^2 + y^2)^{-2}, \frac{\partial}{\partial x} (x^2 + y^2) = - \frac{2y}{(x^2 + y^2)^2}. 2x$$

$$= - \frac{4xy}{(x2 + y2)^2}$$

$$\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial}{\partial y} \left| \frac{\partial u}{\partial x} \right| = \frac{\partial}{\partial y} \left[ 2x (x^2 + y^2)^{-1} \frac{-1}{x} \right]$$

$$= -2x (x^2 + y^2)^{-2.} \frac{\partial}{\partial y} (x^2 + y^2) = \frac{2x}{(x^2 + y^2)^2} . 2y$$

$$= - \frac{4xy}{(x^2 + y^2)^2}$$

$$\therefore \frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$$

(iii) $u = \tan^{-1}\left|\frac{x}{y}\right|$

$$\frac{\partial u}{\partial x} = \frac{1}{1+\left|\frac{x}{y}\right|^2}\ \frac{\partial}{\partial x}\left|\frac{x}{y}\right| = \frac{y2}{(x^2+y^2)}\ \frac{1}{y} = \frac{y}{x^2+y^2}$$

$$\frac{\partial u}{\partial y} = \frac{1}{1+\left|\frac{x}{y}\right|^2}\ \frac{\partial}{\partial y}\left|\frac{x}{y}\right| = \frac{y2}{(x^2+y^2)}\left|-\frac{x}{y^2}\right| = \frac{y}{x^2+y^2}$$

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial x}\left|\frac{\partial u}{\partial y}\right| = \frac{(x^2+y^2)\frac{\partial}{\partial x}(x) - x\frac{\partial}{\partial x}(x^2+y^2)}{(x^2+y^2)^2}$$

$$= \frac{-(x^2+y^2)\quad 1 - x.\ 2x}{(x^2+y^2)^2} = \frac{x^2-y^2}{(x^2+y^2)^2}$$

$$\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial}{\partial y}\left|\frac{\partial u}{\partial x}\right| = \frac{(x^2+y^2)\frac{\partial}{\partial y}(y) - y\frac{\partial}{\partial y}(x^2+y^2)}{(x^2+y^2)^2}$$

$$= \frac{(x^2+y^2)\quad 1 - y.\ 2y}{(x^2+y^2)^2} = \frac{x^2-y^2}{(x^2+y^2)^2}$$

$$\therefore \quad \frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$$

**Example 6.** If $z = \log(e^x + e^y)$, show that $rt - s^2 = 0$

Where $r\ \frac{\partial^2 z}{\partial x^2}$ , $t = \frac{\partial^2 z}{\partial y^2}$ , $s = \frac{\partial^2 z}{\partial x \partial y}$

Sol . $z = \log(e^x + e^y)$

$$\Rightarrow \quad \frac{\partial z}{\partial x} = \frac{1}{e^x+e^y}\ \frac{\partial}{\partial x}(e^x+e^y), \frac{e^x}{e^x+e^y}$$

$$\frac{\partial z}{\partial y} = \frac{1}{e^x+e^y}\ \frac{\partial}{\partial y}(e^x+e^y) = \frac{e^y}{e^x+e^y}$$

$$\therefore \quad r = \frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left|\frac{e^x}{e^x+e^y}\right|$$

$$\frac{(e^x+e^y).\ \frac{\partial}{\partial x}(e^x) - e^x\frac{\partial}{\partial x}(e^x+e^y)}{(e^x+e^y)^2}$$

$$\frac{(e^x+e^y)\quad e^x \ - e^x \cdot e^x}{(e^x+e^y)^2} = \frac{e^{x+y}}{(e^x+e^y)^2}$$

By symmetry,

$$t = \frac{\partial^2 z}{\partial y^2} = \frac{e^{x+y}}{(e^x + e^y)^2} = r$$

$$s = \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left|\frac{\partial z}{\partial y}\right| = \frac{\partial}{\partial x}[e^y (e^x + e^y)^{-1}]$$

$$= e^y . (-1)(e^x + e^y)^{-2} \frac{\partial}{\partial x}(e^x)$$

$$= -\frac{e^{x+y}}{(e^x + e^y)^2} = -r$$

$$\therefore \quad rt - s^2 = r.\ r - (-r)^2 = 0$$

**Example 7.** If $u = x^y$, show that

$$\frac{\partial^3 u}{\partial x^2 \partial y} \quad \frac{\partial^3 u}{\partial x \partial y \partial x}$$

Sol.

$$u = x^y$$

$$\frac{\partial u}{\partial y} = x^y \log x$$

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial x}\left|\frac{\partial u}{\partial y}\right| = yx^{y-1} \log x + x^y . \frac{1}{x}$$

$$= x^{y-1} (y \log x + 1)$$

$$\frac{\partial^3 u}{\partial x^2 \partial y} = \frac{\partial}{\partial x}\left|\frac{\partial^2 x}{\partial x \partial y}\right| = \frac{\partial}{\partial x}[x^{y-1} (y \log x + 1)] \quad ....(1)$$

$$\frac{\partial u}{\partial x} = yx^{y-1}$$

$$\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial}{\partial y}\left|\frac{\partial u}{\partial x}\right| = x^{y-1 + y} x^{y-1} \log x$$

$$= x^{y-1} (y \log x + 1)$$

$$\frac{\partial^3 u}{\partial x \partial y \partial x} = \frac{\partial}{\partial x} \frac{\partial^2 u}{\partial y \partial x} = \frac{\partial}{\partial x}[x^{y-1}(y \log x + 1)] \quad ....(2)$$

From (1) and (2), $\frac{\partial^3 u}{\partial x^2 \partial y} = \frac{\partial^3 u}{\partial x \partial y \partial x}$

**Example 8.** If $x = e^{r\cos\theta}\cos(r\sin\theta)$ and

$y = e^{r\cos\theta}\, r\sin\theta$ prove that $\frac{\partial x}{\partial r} = \frac{1}{r}\frac{\partial y}{\partial \theta}$, $\frac{\partial y}{\partial r} = -\frac{1}{r}\frac{\partial x}{\partial \theta}$

Hence deduce that $\frac{\partial^2 x}{\partial r^2} + \frac{1}{r}\frac{\partial x}{\partial r} + \frac{1}{r^2}\frac{\partial^2 x}{\partial \theta^2} = 0$

Sol. $x = e^{r\cos\theta} . \cos(r\sin\theta)$

$$\therefore \frac{\partial x}{\partial r} = e^{r\cos\theta} . \cos\theta) .\cos(r\sin\theta) - e^{r\cos\theta}\sin(r\sin\theta) . \sin\theta$$

$$= e^{r\cos\theta}[\cos\theta\cos(r\sin\theta) - \sin\theta\sin(r\sin\theta)]$$

$$= e^{r\cos\theta}\cos(\theta + r\sin\theta)$$

$$\frac{\partial x}{\partial \theta} = e^{r\cos\theta}(-r\sin\theta)\cos(r\sin\theta)$$

$$-e^{r\cos\theta}\sin(r\sin\theta)\; r\cos\theta)$$

$$= -re^{r\cos\theta}[\sin\theta\cos(r\sin\theta) + \cos\theta\sin(r\sin\theta)]$$

$$= -re^{r\cos\theta}\sin(\theta + r\sin\theta)$$

Also $y = e^{r\cos\theta}\sin(r\sin\theta)$

$$\therefore \frac{\partial y}{\partial r} = e^{r\cos\theta} .\cos\theta . \sin(r\sin\theta) + e^{r\cos\theta} . \cos(r\sin\theta)\sin\theta$$

$$= e^{r\cos\theta}(\sin\theta\cos(r\sin\theta) + \cos\theta\sin(r\sin\theta))$$

$$= e^{r\cos\theta}\sin(\theta + r\sin\theta)$$

$$\frac{\partial y}{\partial \theta} = e^{r\cos\theta}(-r\sin\theta)\sin(r\sin\theta)$$

$$+e^{r\cos\theta}\cos(r\sin\theta) . r\cos\theta$$

$$= re^{r\cos\theta}[\cos\theta\cos(r\sin\theta) - \sin\theta\sin(r\sin\theta)]$$

$$= re^{r\cos\theta}\cos(\theta + r\sin\theta)$$

From (1) and (4), $\frac{\partial x}{\partial r} = \frac{1}{r}\frac{\partial y}{\partial \theta}$

From (2) and (3), = $\frac{\partial y}{\partial r} = -\frac{1}{r}\frac{\partial x}{\partial \theta}$

From (5), $\frac{\partial^2 x}{\partial r^2} = \frac{-1}{r^2}\frac{\partial y}{\partial \theta} + \frac{1}{r}\frac{\partial^2 y}{\partial r\partial \theta}$

From (6) $\quad \dfrac{\partial x}{\partial \theta} = -r \dfrac{\partial y}{\partial r}$

$$\therefore \quad \frac{\partial^2 x}{\partial \theta^2} = -r \frac{\partial^2 y}{\partial \theta \partial r} = -r \frac{\partial^2 y}{\partial r \partial \theta}$$

$$\therefore \quad \frac{\partial^2 x}{\partial r^2} + \frac{1}{r} \cdot \frac{\partial x}{\partial r} + \frac{1}{r^2} \cdot \frac{\partial^2 x}{\partial \theta^2}$$

$$= -\frac{1}{r^2} \cdot \frac{\partial y}{\partial \theta} + \frac{1}{r} \cdot \frac{\partial^2 y}{\partial r \partial \theta} + \frac{1}{r^2} \cdot \frac{\partial y}{\partial \theta} - \frac{1}{r} \cdot \frac{\partial^2 y}{\partial r \partial \theta} = 0$$

## 3.4. HOMOGENEOUS FUNCTIONS

A function $f(x, y)$ is said to be homogeneous of degree (or order) n in the variables x and y if it can be expressed in the form $x^n \phi\left[\frac{y}{x}\right]$ or $y^n \phi\left[\frac{x}{y}\right]$

An alternative test for a function $f(x, y)$ to be homogeneous of degree (or order) n is that $f(tx, ty) = t^n f(x, y)$

For example, if $f(x, y) = \dfrac{x+y}{\sqrt{x}+\sqrt{y}}$, then

(i) $f(x, y) = \dfrac{x\left|1+\frac{y}{x}\right|}{\sqrt{x}\left|1+\sqrt{\frac{y}{x}}\right|} = x^{1/2} \phi\left[\frac{y}{x}\right]$

⇨ $f(x, y)$ is a homogeneous function of degree 1/2 in x and y.

(ii) $f(x, y) = \dfrac{y\left|\frac{x}{y}+1\right|}{\sqrt{y}\left|\sqrt{\frac{x}{y}}+1\right|} = y^{1/2} \phi\left[\frac{x}{y}\right]$

⇨ $f(x, y)$ is a homogeneous function of degree 1/2 in x and y.

(iii) $f(tx, ty) = \dfrac{tx+ty}{\sqrt{tx}+\sqrt{ty}} = \dfrac{t(x+y)}{\sqrt{t}\,(\sqrt{x}+\sqrt{y})} = t^{1/2} f(x, y)$

⇨ $f(x, y)$ is a homogeneous function of degree 1/2 in x and y.

Similarly, a function f(x, y, z)is said to be homogeneous of degree (or order) n in the variables x, y, z if

$f(x, y, z) = x^n \phi\left[\frac{y}{x} \ \frac{z}{x}\right]$ or $y^n \phi\left[\frac{x}{y} \ \frac{z}{y}\right]$ or $z^n \phi\left[\frac{x}{z} \ \frac{y}{z}\right]$

Alternative test is $f(tx, ty, tz) = t^n f(x, y, z)$

## 3.5. EULER'S THEOREM HOMOGENEOUS FUNCTIONS

If u is a homogeneous function of degree n in x and y, then

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial x} = nu$$

Since u is a homogeneous function of degree n in x and y, it can be expressed as

$$u = x^n f\left[\frac{y}{x}\right]$$

$$\therefore \frac{\partial u}{\partial x} = nx^{n-1} f\left[\frac{y}{x}\right] + x^n f'\left[\frac{y}{x}\right] \left[\frac{-y}{x^2}\right]$$

$$\Rightarrow x \frac{\partial u}{\partial x} = nx^n f \left[\frac{y}{x}\right] - x^{n-1} y f' \left[\frac{y}{x}\right]$$

Also $\frac{\partial u}{\partial y} = x^n f' \left[\frac{y}{x}\right] \cdot \frac{1}{x} = x^{n-1} f' \left[\frac{y}{x}\right]$

$$\Rightarrow y \frac{\partial u}{\partial y} = x^{n-1} y f' \left[\frac{y}{x}\right]$$

Adding (1) and (2), we get

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial x} = nx^n f \frac{y}{x} = nu$$

Note : Euler's theorem can be extended to a homogeneous function of any number of variables. Thus, if u is a homogeneous function of degree n in x, y and z then

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial x} + z \frac{\partial u}{\partial x} = nu$$

## 3.6. If u is a homogeneous function of degree n in x and y, then

$$x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = n(n-1) u$$

Since u is a homogeneous function of degree n in x and y

∴ By Euler's theorem, we have

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = nu$$

Differentiating (1) partially w.r.t.x, we have

$$1. \frac{\partial u}{\partial x} + x. \frac{\partial^2 u}{\partial x^2} + y. \frac{\partial^2 u}{\partial x \partial y} = n \frac{\partial u}{\partial x}$$

Differentiating (1) partially w.r.t.y, we have

$$x \frac{\partial^2 u}{\partial y \partial x} + 1 \cdot \frac{\partial u}{\partial y} + y \frac{\partial^2 u}{\partial y^2} = n \frac{\partial u}{\partial y}$$

But $\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial^2 u}{\partial x \partial y}$

$\therefore$ $x \frac{\partial^2 u}{\partial x \partial y} + \frac{\partial u}{\partial y} + y \frac{\partial^2 u}{\partial y^2} = n \frac{\partial u}{\partial y}$

Multiplying (2) by x, (3) by y and adding

$$x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial u^2} + \left| x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} \right|$$

$$= n + \left| x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} \right|$$

or $$x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} + nu = n.nu$$

$\Rightarrow$ $$x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = n^2 u - nu = n(n-1)u$$

Note . If $f$ (x, y) is homogeneous n, show that

$$x^2 f11 + xyf12 + xyf21 + y^2 f22 = n(n-1)f$$

## ILLUSTRATIVE EXAMPLES

**Example 1.** Verify Euler's theorem for the functions:

(i) $u = (x^{½} + y^{½})(x^n + y^n)$ ii $u = \sin^{-1} \frac{x}{y} + \tan^{-1} \frac{y}{x}$

**Sol**. i. $u = (x^{½} + y^{½})(x^n + y^n)$

$$= (x^{½} (1 + \frac{y^{½}}{x^{½}})\ x^n (1 + \frac{y^n}{x^n})$$

$$= x^{n½} \left[ 1 + \left(\frac{y}{x}\right)^{½} \right] \left[ 1 + \left(\frac{y}{x}\right)^n \right] = x^{n+½} f\left(\frac{y}{x}\right)$$

[OR $f(tx, ty) = t^{n+½} f(x, y)$]

$\Rightarrow$ u is a homogeneous function of degree (n+½) in x and y.

$\therefore$ By Euler's theorem, we should have

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = (n+½)u$$

From (1), $\frac{\partial u}{\partial x} = \frac{1}{2} x^{-\frac{1}{2}} (x^n + y^n) + nx^{n-1} (x^{\frac{1}{2}}+y^{\frac{1}{2}})$

$$x \frac{\partial u}{\partial x} = \frac{1}{2} x (x^n + y^n) + nx^n (x^{\frac{1}{2}}+y^{\frac{1}{2}})$$

$$\frac{\partial u}{\partial y} = \frac{1}{2} y^{-\frac{1}{2}} (x^n + y^n) + ny^{n-1} (x^{\frac{1}{2}}+y^{\frac{1}{2}})$$

$$y \frac{\partial u}{\partial x} = \frac{1}{2} y^{\frac{1}{2}} (x^n + y^n) + ny^n (x^{\frac{1}{2}}+y^{\frac{1}{2}})$$

$$\therefore \quad x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = \frac{1}{2} (x^{\frac{1}{2}}+y^{\frac{1}{2}}) (x^n + y^n) + n(x^n + y^n) (x^{\frac{1}{2}}+y^{\frac{1}{2}})$$

$$= \frac{1}{2} u + nu = (n+\tfrac{1}{2})u$$

Which is the same as (2). Hence the verification.

(ii) $u = \sin^{-1} \frac{x}{y} + \tan^{-1} \frac{y}{x}$

$$= \text{cosec}^{-1} \frac{y}{x} + \tan^{-1} \frac{y}{x} = x^0 f \left[ \frac{y}{x} \right]$$

[OR $f(tx, ty) = f(x, y) = t^0 f(x, y)$)

⇨ u is a homogeneous function of degree 0 in x and y.

∴ By Euler's theorem, we should have

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = 0 \text{ x } u = 0$$

From (1), $\frac{\partial u}{\partial x} = \frac{1}{\sqrt{1-\frac{x^2}{y^2}}} \cdot \frac{1}{y} + \frac{1}{1+\frac{y^2}{x^2}} - \left| \frac{y}{x^2} \right|$

$$= \frac{1}{\sqrt{y^2-x^2}} - \frac{y}{x^2+y^2}$$

$$x \frac{\partial u}{\partial x} = \frac{x}{\sqrt{y2-x2}} - \frac{xy}{x2+y2}$$

$$\frac{\partial u}{\partial y} = \frac{1}{\sqrt{1-\frac{x^2}{y^2}}} \left| -\frac{x}{y^2} \right| + \frac{1}{\sqrt{1+\frac{y^2}{x^2}}} \cdot \frac{1}{x}$$

$$-y \quad \frac{x}{\sqrt{y^2-x^2}} + \frac{x}{x^2+y^2}$$

$$y\frac{\partial u}{\partial y} = \frac{x}{\sqrt{y^2-x^2}} + \frac{xy}{x^2+y^2}$$

$$\therefore \quad x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0$$

Which is the same as (2). Hence the verification.

## 3.7 COMPOSITE FUNCTIONS

(i) If $u = f(x,y)$ where $x = \phi(t)$, $y = \psi(t)$
Then u is called a composite function of (the single variable) t and we can find $\frac{du}{dt}$

(ii) If $z = f(x, y)$ where $x = \phi(u, v)$, $y = \psi(u, v)$
Then z is called a composite function of (two variables) u and v so that we can find $\frac{\partial z}{\partial u}$ and $\frac{\partial z}{\partial v}$

## 3.8 DIFFERENTIATION OF COMPOSITE FUNCTIONS

If u is a composite function of t, defined by the relations
$u = f(x,y)$; $x = \phi(t)$, $y = \psi(t)$, then

$$\frac{du}{dt} = \frac{\partial u}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dt}$$

[ $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$ $\therefore$ u is a function of two variables x and y ]

[ $\frac{dx}{dt}$ and $\frac{dy}{dt}$ $\therefore$ x and y are function of a single variable t ]

## ILLUSTRATIVE EXAMPLES

**Example 1** . Find $\frac{du}{dt}$ when $u = xy^2 + x^2y$, $x = at^2$, $y = 2at$.

Also verify by direct substitution

**Sol.** the given equations define u as a composite function of t.

$$\therefore \frac{du}{dt} = \frac{\partial u}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dt}$$

$= (y^2+2xy).\, 2at + (2xy+x^2).2a$

$= (4a^2t^2+2at^2 \cdot 2at)\, 2at + (2at^2.2at+a^2t^4).2a$

$= 8a^3t^3+8a^3t^4 + 8a^3t^3+2a^3t^4$

$= 2a^3t^3(5t+8)$

Also $u = xy^2+x^2y = at^2.4a^2t^2+a^2t^4 \cdot 2at$

$= 4a^3t^4 + 2a^3t^5$

$\frac{du}{dt} = 16a^3t^3 + 10a^3t^4 = 2a^3t^3\,(5t+8)$

Hence the verification

Example 2. Find $\frac{d^2y}{dx^2}$ If $ax^2 + 2hxy + by^2 = 1$

Sol. Let $f(x, y) = ax^2 + 2hxy + by^2 - 1$

$f_x = 2ax + 2hy, \quad f_y = 2hx + 2by$

$f_{xx} = 2a, \ f_{xy} = 2h, \qquad f_{yy} = 2b$

$$\therefore \quad \frac{d^2y}{dx^2} = \frac{f_{xx}f_y^2 - 2f_xf_yf_{xy} + f_{yy}f_x^2}{f_y^3}$$

$$= -\frac{2a(2hx+2by)^2 - 2(2ax+2hy)(2hx+2by)(2h) + 2b(2ax+2hy)^2}{(2hx+2by)^3}$$

$$= -\frac{a(hx+by)^2 - 2h(ax+hy)(hx+by) + b(ax+hy)^2}{(hx+by)^3}$$

$$= \frac{(ah^2 - 2ah^2 + ba^2)x^2 + (2abh - 2abh - 2h^3 + 2abh)xy + (ab^2 - 2bh^2 + bh^2)y^2}{(hx+by)^3}$$

$$= -\frac{a(ab - h^2)x^2 + 2h(ab - h^2)xy + b(ab - h^2)y^2}{(hx+by)^3}$$

$$= \frac{(h^2 - ab)(ax^2 + 2xhy + by^2)}{(hx+by)^3}$$

$$= \frac{h^2 - ab}{(hx+by)^3}$$

$[\because ax^2 + 2hxy + by^2 = 1]$

**H.W**

1. Find $\frac{du}{dt}$ when $u = x^2+y^2$, $x=at^2$, $y=2at$. Also verify by direct substitution.

Ans: $4a^2t(t^2+2)]$

2. If $u = \sin\frac{x}{y,}$ $x = e^t$, $y = t^2$ find $\frac{du}{dt}$

Ans: $\frac{e^t(t-2)}{t^3} \cos\left|\frac{e^t}{t^2}\right|$

3. If $u = x^3+y^3$, where $x = a \cos t$, $y= b \sin t$, find $\frac{du}{dt}$ and verify the result.

Ans: $-3a^3\cos^2 t \sin t +3b^3 \sin^2 t \cos t$

4. If $z = u^2 + v^2$, $u = r \cos \phi$, $vr \sin \phi$, find $\frac{\partial z}{\partial r}$ and $\frac{\partial z}{\partial \phi}$

Ans: 2r, 0

5. Given $v = f(x, y, z)$, $x = r \cos \phi$, $y = r \sin \phi$, $z = t$, obtain expressions for

$\frac{\partial v}{\partial r}, \frac{\partial v}{\partial \phi}, \frac{\partial v}{\partial t}$ in terms of $\frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}, \frac{\partial v}{\partial z}$

Ans: $\frac{\partial v}{\partial r} = \frac{\partial v}{\partial x} \cos \phi + \frac{\partial v}{\partial y} \sin \phi$ ; $\frac{\partial v}{\partial \phi} = -\frac{\partial v}{\partial x} .r \sin \phi + \frac{\partial v}{\partial y} .r \cos \phi$; $\frac{\partial v}{\partial t} = \frac{\partial v}{\partial z}$

6. If $u = f(r, s)$, $r = x + y$, $s = x-y$, show that

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 2\frac{\partial u}{\partial x}$$

7. If $x = u+v$, $y = uv$ and z is a function of x, y show that

$$u\frac{\partial z}{\partial u} + v\frac{\partial z}{\partial v} = x\frac{\partial z}{\partial x} + 2y\frac{\partial z}{\partial y}$$

8. If $u = f(r, s)$, $r = x+at$, $s = y+bt$, show that

$$\frac{\partial u}{\partial t} = a\frac{\partial u}{\partial x} + b\frac{\partial u}{\partial y}$$

9. If $u = e^{mx}(y-z)$, $y = m \sin x$ and $z = \cos x$, find $\frac{du}{dx}$

Ans: $e^{mx}(m^2+1) \sin x$

10. If $u = x \log (xy)$, where $x^3+y^3+3xy = 1$, find $\frac{du}{dx}$

Ans: $1+\log (xy) - \frac{x(x^2+y)}{y(y^2+x)}$

11. (i) If $u = f(r, s, t)$ and $r = \frac{x}{y}$, $s = \frac{y}{z}$, $t = \frac{z}{x}$, prove that

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 0$$

(ii) If $x = \sqrt{vw}$ , $y = \sqrt{uw}$, $z = \sqrt{uv}$ and $\phi$ is a function of x, y, z, then

$$x\frac{\partial \phi}{\partial x} + y\frac{\partial \phi}{\partial y} + z\frac{\partial \phi}{\partial z} = u\frac{\partial \phi}{\partial u} + v\frac{\partial \phi}{\partial v} + w\frac{\partial \phi}{\partial w}$$

12. If $z = x^2y$ and $x^2 + xy + y^2 = 1$, show that

$$\frac{dz}{dx} = 2xy - \frac{x^2(2x+y)}{x+2y}$$

13. If $x^3 + 3x^2y + 6xy^2 + y^3 = 1$, find $\frac{dy}{dx}$

Ans: $-\frac{x^2 + 2xy + 2y^2}{x^2 + 4xy + y^2}$

14. If $x^y = y^x$ show that $\frac{dy}{dx} = \frac{y(y - x\log y)}{x(x - y\log x)}$

using partial derivative method

15. Find $\frac{dy}{dx}$ from (i) $xy\log(x+y) = 1$ (ii) $xy = e^{x^2+y^2}$

Ans: (i) $\frac{y(x+y)\log(x+y)\, xy}{x(x+y)\log(x+y) + xy}$, (ii) $-\frac{y - 2xe^{x^2+y^2}}{x - 2ye^{x^2+y^2}}$

16. Prove that if $y^3 - 3ax^2 + x^3 = 0$, then $\frac{d^2y}{dx^2} + \frac{2a^2x^2}{y^5} = 0$

17. By changing the independent variables x and y to u and v by means of the relations $u = x - ay$, $v = x + ay$, show that

$$a^2\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial y^2} \text{ transforms in to } 4a^2\frac{\partial^2 z}{\partial u \partial v}$$

18. If z is a function of x and y, and u and v be other variables sucha that $u = lx + my$, $v = ly - mx$, show that $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = (l^2 + m^2)\left(\frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial v^2}\right)$

19. If $u = u(x, y)$ and $x = e^r \cos\phi$, $y = e^r \sin\phi$, show that

(i) $$\left[\frac{\partial u}{\partial x}\right]^2 + \left[\frac{\partial u}{\partial y}\right]^2 = e^{-2r}\left[\left[\frac{\partial u}{\partial r}\right]^2 + \left[\frac{\partial u}{\partial \phi}\right]^2\right]$$

(ii) $$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = e^{-2r}\left[\frac{\partial^2 u}{\partial r^2} + \frac{\partial^2 u}{\partial \phi^2}\right]$$

20. (i) If $v = f(x-y, y-z, z-x)$, show that $\frac{\partial v}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial v}{\partial z} = 0$

(ii) If $u = f(x^2-y^2, y^2-z^2, z^2-x^2)$, prove that $\frac{1}{x}\frac{\partial u}{\partial x} + \frac{1}{y}\frac{\partial u}{\partial y} + \frac{1}{z}\frac{\partial u}{\partial z} = 0$

21. If $x = \tilde{n} \cos\phi$, $y = \tilde{n} \sin\phi$, show that

$$\left[\frac{\partial v}{\partial x}\right]^2 + \left[\frac{\partial v}{\partial y}\right]^2 = \left[\frac{\partial v}{\partial \tilde{n}}\right]^2 + \frac{1}{\tilde{n}^2}\left[\frac{\partial v}{\partial \phi}\right]^2$$

22. Prove that $$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial s^2} + \frac{\partial^2 u}{\partial t^2}$$

Where $x = s \cos a - t \sin a$. and $y = s \sin a + t \cos a$

23. If $z = uv$, $u^2+v^2-x-y=0$, $u^2-v^2+3x+y=0$, find $\frac{\partial z}{\partial x}$ Ans. $\frac{2u^2-v^2}{2uv}$

24. If $x = u^2-v^2$, $y = 2uv$, find $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$, $\frac{\partial v}{\partial x}$, $\frac{\partial v}{\partial y}$

Ans: $\frac{\partial u}{\partial x} = \frac{u}{2(u^2+v^2)}$, $\frac{\partial u}{\partial y} = \frac{v}{2(u^2+v^2)}$

$\frac{\partial v}{\partial x} = \frac{-v}{2(u^2+v^2)}$ = $\frac{\partial v}{\partial y} = \frac{u}{2(u^2+v^2)}$

## 3.9. JACOBIANS

If u and v are functions of two independent variables x and y, then the determinant

$$\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$

is called the Jacobian of u, v with respect to x, y and is denoted by the symbol

$$J \left| \frac{u,\ v}{x,\ y} \right| \quad \text{or} \quad \frac{\partial\ (u,\ v)}{\partial\ (x,\ y)}$$

Similarly, if u, v, w be functions of x, y, z then the Jacobian of u, v, w with respect to x, y, z is

$$J \left| \frac{u,\ v,\ w}{x,\ y,\ z} \right| \quad \text{or} \quad \frac{\partial\ (u,\ v,\ w)}{\partial\ (x,\ y,\ z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$$

## 3.10. PROPERTIES OF JACOBIANS

I. If u, v are functions of r, s where r, s are functions of x, y, then

$$\frac{\partial\ (u,\ v)}{\partial\ (x,\ y)} = \frac{\partial\ (u,\ v)}{\partial\ (r,\ s)} \times \frac{\partial\ (r,\ s)}{\partial\ (x,\ y)}$$

Proof. Since u, v are composite functions of x, y

$$\therefore \quad \left.\begin{aligned} \frac{\partial u}{\partial x} &= \frac{\partial u}{\partial r} \cdot \frac{\partial r}{\partial x} + \frac{\partial u}{\partial s} \cdot \frac{\partial s}{\partial x} = u_r r_x + u_s s_x \\ \frac{\partial u}{\partial y} &= \frac{\partial u}{\partial r} \cdot \frac{\partial r}{\partial y} + \frac{\partial u}{\partial s} \cdot \frac{\partial s}{\partial y} = u_r r_y + u_s s_y \\ \frac{\partial v}{\partial x} &= \frac{\partial v}{\partial r} \cdot \frac{\partial r}{\partial x} + \frac{\partial v}{\partial s} \cdot \frac{\partial s}{\partial x} = v_r r_x + u_s s_x \\ \frac{\partial v}{\partial y} &= \frac{\partial v}{\partial r} \cdot \frac{\partial r}{\partial y} + \frac{\partial v}{\partial s} \cdot \frac{\partial s}{\partial y} = v_r r_y + u_s s_y \end{aligned}\right| \quad \text{....A}$$

$$\text{Now} \quad \frac{\partial\ (u,\ v)}{\partial\ (r,\ s)} \times \frac{\partial\ (r,\ s)}{\partial\ (x,\ y)} = \begin{vmatrix} u_r, & u_s \\ v_r, & v_s \end{vmatrix} \cdot \begin{vmatrix} r_x & r_y \\ s_x & s_y \end{vmatrix}$$

Interchanging rows and columns in the second determinant

$$= \begin{vmatrix} u_r, & u_s \\ v_r, & v_s \end{vmatrix} \cdot \begin{vmatrix} r_x & s_x \\ r_y & s_y \end{vmatrix}$$

$$= \begin{vmatrix} u_r r_x + u_s s_x & u_r r_y + u_s s_y \\ v_r r_x + v_s s_x & v_r r_y + v_s s_y \end{vmatrix}$$

$$= \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$

$$= \frac{\partial\ (u,\ v)}{\partial\ (x,\ y)}$$

II. If $J_1$ is the Jacobian of u, v with respect to x, y and $J_2$ is the Jacobian of x, y with respect to u, v then $J_1, J_2 = 1$

i.e, $$\frac{\partial\ (u,\ v)}{\partial\ (x,\ y)} \cdot \frac{\partial\ (x,\ y)}{\partial\ (u,\ v)} = 1$$

**Proof.** Let u = u(x, y) and v = v(x, y), so that u and v are functions of x, y.

Differentiating partially w.e.t.u and v, we get

$$\left.\begin{aligned} 1 &= \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial u} = u_x x_u + u_y y_u \\ 0 &= \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial v} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial v} = u_x x_v + u_y y_v \\ 0 &= \frac{\partial v}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial v}{\partial y} \cdot \frac{\partial y}{\partial u} = v_x x_u + v_y y_u \\ 1 &= \frac{\partial v}{\partial x} \cdot \frac{\partial x}{\partial v} + \frac{\partial v}{\partial y} \cdot \frac{\partial y}{\partial v} = v_x x_v + v_y y_v \end{aligned}\right| \text{....A}$$

Now $$\frac{\partial\ (u,\ v)}{\partial\ (x,\ y)} \cdot \frac{\partial\ (x,\ y)}{\partial\ (u,\ v)} = \begin{vmatrix} u_x, & u_y \\ v_x, & v_y \end{vmatrix} \cdot \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix}$$

Interchanging rows and columns in the second determinant

$$= \begin{vmatrix} u_x, & u_y \\ v_x, & v_y \end{vmatrix} \cdot \begin{vmatrix} x_u & y_u \\ x_v & y_v \end{vmatrix}$$

$$= \begin{vmatrix} u_x x_u + u_y y_u & u_x x_v + u_y y_v \\ v_x x_u + v_y y_u & v_x x_v + v_y y_v \end{vmatrix} \qquad \text{[Using (A)]}$$

$$= \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$$

**ILLUSTRATIVE EXAMPLES**

**Example 1.** If x = r cos ϕ, y = r sin ϕ , prove that the Jacobians

$$J = \frac{\partial (x, y)}{\partial (r, \phi)} = r \text{ and } J = \frac{\partial (r, \phi)}{\partial (x, y)} = \frac{1}{J}$$

[A.M.I.E (S) 1993; (S) 94; (S) 95; (W) 97]

**Sol.** $x = r\cos\phi$, $y = r\sin\phi$

$$\therefore \quad \frac{\partial x}{\partial r} = \cos\phi \qquad \frac{\partial x}{\partial \phi} = -r\cos\phi$$

$$\frac{\partial y}{\partial r} = \sin\phi, \qquad \frac{\partial y}{\partial \phi} = r\cos\phi$$

$$\therefore \quad J = \frac{\partial (x, y)}{\partial (r, \phi)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \phi} \end{vmatrix} = \begin{vmatrix} \cos\phi & -r\sin\phi \\ \sin\phi & r\cos\phi \end{vmatrix}$$

$$= r\cos^2\phi + r\sin^2\phi = r$$

Now $x^2 + y^2 = r^2$ and $\frac{y}{x} = \tan\phi$

$$\Rightarrow \quad r = \sqrt{x^2 + y^2} \quad \text{and} \quad \phi = \tan^{-1}\frac{y}{x}$$

$$\frac{\partial r}{\partial x} \; \frac{x}{\sqrt{x^2 + y^2}}, \quad \frac{\partial r}{\partial y} = \frac{y}{\sqrt{x^2 + y^2}}$$

$$\frac{\partial \phi}{\partial x} = \frac{1}{1 + \frac{y^2}{x^2}} \frac{(-y)}{x^2} = -\frac{y}{x^2 + y^2}$$

$$\frac{\partial \phi}{\partial y} \; \frac{1}{1 + \frac{y^2}{x^2}} \left(\frac{1}{x}\right) = \frac{x}{x^2 + y^2}$$

$$\therefore \quad J = \frac{\partial (r, \phi)}{\partial (x, y)} = \begin{vmatrix} \frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} \\ \frac{\partial \phi}{\partial x} & \frac{\partial \phi}{\partial y} \end{vmatrix} = \begin{vmatrix} \frac{x}{\sqrt{x^2 + y^2}} & \frac{y}{\sqrt{x^2 + y^2}} \\ \frac{-y}{\sqrt{x^2 + y^2}} & \frac{x}{\sqrt{x^2 + y^2}} \end{vmatrix}$$

$$= \frac{x^2}{(x^2 + y^2)^{3/2}} + \frac{y^2}{(x^2 + y^2)^{3/2}} = \frac{x^2 + y^2}{(x^2 + y^2)^{3/2}}$$

$$= \frac{1}{\sqrt{x^2 + y^2}} = \frac{1}{r} = \frac{1}{J}$$

## 3.11. TAYLOR'S THEOREM FOR A FUNCTION OF TWO VARIABLES

We know that by Taylor's theorem for a function *f*(x) of ssingle variable x,

$$f(x+h)=f(x) + hf'(x)+ \frac{h^2}{2!} f''(x) + \frac{h^3}{3!} f'''(x) + .....$$

Now let *f*(x, y) be a function of two independent variables x and y. If y is kept constant, then by Taylor's theorem for a function of a single variable x, we have

$$f(x+h, y+k) = f(x, y+k) +h \frac{\partial}{\partial x} f(x, y+k) + \frac{h^2}{2!}\frac{\partial^2}{\partial x^2} f(x, y+k) + \frac{h^3}{3!}\frac{\partial^3}{\partial x^3} f(x, y+k)+... \quad ..(1)$$

Now keeping x constant and applying Taylor;s Theorem for a function of a single variable y, we have

$$f(x, y+k) = f(x, y) +k \frac{\partial}{\partial y} f(x, y) + \frac{k^2}{2!}\frac{\partial^2}{\partial y^2} f(x, y) + \frac{k^3}{3!}\frac{\partial^3}{\partial y^3} f(x, y)+... \quad .(2)$$

Using (2), we can write (1) as

$$f(x+h, y+k) = [f(x, y) +k \frac{\partial}{\partial y} f(x, y) + \frac{k^2}{2!}\frac{\partial^2}{\partial y^2} f(x, y) + \frac{k^3}{3!}\frac{\partial^3}{\partial y^3} f(x+y) + ......]$$

$$+ h\frac{\partial}{\partial x}[f(x, y) +k \frac{\partial}{\partial y} f(x, y) + \frac{k^2}{2!}\frac{\partial^2}{\partial y^2} f(x, y) + ....]$$

$$+ \frac{h^2}{2!}\frac{\partial^2}{\partial x^2} [f(x, y) +k\frac{\partial}{\partial y} f(x, y) + ....]$$

$$+ \frac{h^3}{3!}\frac{\partial^3}{\partial x^3} [f(x, y) + ..........] + .......$$

$$= [f(x, y) + \left(h \frac{\partial f}{\partial x} + k \frac{\partial f}{\partial y}\right) + \left(\frac{h^2}{2!}\frac{\partial^2 f}{\partial x^2} + hk\frac{\partial^2 f}{\partial x \partial y} + \frac{k^2}{2!}\frac{\partial^2 f}{\partial y^2}\right)$$

$$+ \left(\frac{h^3}{3!}\frac{\partial^3 f}{\partial x^3} + \frac{h^2 k}{2!}\frac{\partial^3 f}{\partial x^2 \partial y} + \frac{hk^2}{2!}\frac{\partial^3 f}{\partial x \partial y^2} + \frac{k^3}{3!}\frac{\partial^3 f}{\partial y^3}\right) + .......$$

$$= f(x, y) + \left(h \frac{\partial f}{\partial x} + k \frac{\partial f}{\partial y}\right) + \frac{1}{2!}\left(h^2 \frac{\partial^2 f}{\partial x^2} + 2hk\frac{\partial^2 f}{\partial x \partial y} + k^2 \frac{\partial^2 f}{\partial y^2}\right)$$

$$+ \frac{1}{3!}\left(h^3 \frac{\partial^3 f}{\partial x^3} + 3h^2 k \frac{\partial^3 f}{\partial x^2 \partial y} + 3hk^2 \frac{\partial^3 f}{\partial x \partial y^2} + k^3 \frac{\partial^3 f}{\partial y^3}\right) + .......$$

$$= f(x, y) + \left(h \frac{\partial}{\partial x} + k \frac{\partial}{\partial y}\right) + \frac{1}{2!}\left(h \frac{\partial}{\partial x} + k \frac{\partial}{\partial y}\right)^2 f$$

$$+ \frac{1}{3!}\left(h\frac{\partial}{\partial x} + k \frac{\partial}{\partial y}\right)^3 f + .......$$

**Cor. 1**. Putting x = a and y = b, we have

$$f(a+h, b+k) = f(a, b)+[hf_x(a, b) + kf_y(a, b)]$$

$$+ \frac{1}{2!}[h^2 f_{xx}(a, b) + 2hkf_{xy}(a, b) + k^2 f_{yy}(a, b)]$$

$$+ \frac{1}{3!}[h^3 f_{xxx}(a, b) + 3h^2kf_{xxy}(a, b)$$

$$+ 3hk^2 f_{xyy}(a, b) + k^3 f_{yyy}(a, b)] + .......$$

**Cor.2.** In Cor 1 Putting a+h = x and b+k = y so that h = x - a and k = y - b, we have

$$f(x+y) = f(a, b)+[(x-a) f_x(a, b) + (y-b) f_y(a, b)]$$

$$+ \frac{1}{2!}[(x-a)^2 f_{xx}(a, b) + 2(x-a)(y-b) f_{xy}(a, b)$$

$$+ (y-b)^2 f_{yy}(a, b)] + .....$$

**Cor. 3**. Putting a = 0, b = 0 in Core 2, we have

$$f(x, y)= f(0, 0)+[xf_x(0, 0) + yf_y(0, 0)] + \frac{1}{2!}[x^2 f_{xx}(0, 0)$$

$$+ 2xy f_{xy}(0, 0) + y^2 f_{yy}(0, 0)] + .......$$

This is called Maclaurin's theorem for two variables.

[Note. Cor. 3 is used to expand $f(x, y)$ in powers of x and y (or to expand $f(x, y)$ in the neighbourhood of origin (0, 0)].

Cor 2 is used to expand $f(x, y)$ in the neighbourhood of (a, b)].

## ILLUSTRATIVE EXAMPLES

Example 1. Expand $e^x \sin y$ in powers of x and y as far as terms of the third degree.

Sol. Here

| | |
|---|---|
| $f(x, y) = e^x \sin y,$ | $f(0, 0) = 0$ |
| $f_x(x, y) = e^x \sin y,$ | $f_x(0, 0) = 0$ |
| $f_y(x, y) = e^x \cos y,$ | $f_y(0, 0) = 1$ |
| $f_{xx}(x, y) = e^x \sin y,$ | $f_{xx}(0, 0) = 0$ |
| $f_{xy}(x, y) = e^x \cos y,$ | $f_{xy}(0, 0) = 1$ |
| $f_{yy}(x, y) = -e^x \sin y,$ | $f_{yy}(0, 0) = 0$ |
| $f_{xxx}(x, y) = e^x \sin y,$ | $f_{xxx}(0, 0) = 0$ |
| $f_{xxy}(x, y) = e^x \cos y,$ | $f_{xxy}(0, 0) = 0$ |
| $f_{xyy}(x, y) = -e^x \sin y,$ | $f_{xyy}(0, 0) = 0$ |
| $f_{yyy}(x, y) = -e^x \cos y,$ | $f_{yyy}(0, 0) = -1$ |

$\therefore$ ..........................................................................................

$e^x \sin y = f(x, y)$

$= f(0,0) + [x f x(0,0) + y f y(0,0)] + \frac{1}{2!}\,[x^2 f xx(0,0)$

$+ 2xy f xy(0,0) + y^2 f yy(0,0)] + \frac{1}{3!}\,[x^3 f xxx(0,0)$

$+ 3x^2 y f xxy(0,0) + 3xy^2 f xyy(0,0) + y^3 f yyy(0,0)] + ....$

$= 0 + [x.0 + y.1] + \frac{1}{2!}[x^2.0 + 2xy.1 + y^2.0] + \frac{1}{3!}[x^3 \cdot 0$

$+3x^2\, y.1 + 3xy^2\, .0 + y^3\,(-1)] + ........$

$= \quad y + xy + \frac{1}{2!}\,x^2 y - \frac{1}{6}\,y^3 + ......$

**H.W.**

1. Show that $e^x \log(1+x) = x + xy - \frac{x^2}{2}$ approximately.

   [Hint. Find the expansion at (0, 0)]

2. Expand $e^x \cos y$ in powers of x and y as far as the terms of third degree.

   [Ans. $1 + x + \frac{1}{2!}(x^2 - y^2) + \frac{1}{3!}(x^3 - 3xy^2) + .....$]

3. Expand $e^{ax} \sin by$ in powers of x and y as far as the terms of third degree.

   [Ans. $by + abxy + \frac{1}{3!}(3a^2\, bx^2\, y - b^3\, y^3) + ....$]

4. Expand $e^{xy}$ at (1, 1).

   [Ans. $e\{1 + (x-1) + (y-1) + \frac{1}{2!}\left((x-1)^2 + 4(x-1)(y-1) + (y-1)^2\right) + ....\}$]

5. Expand $e^x \cos y$ at $\left(1, \frac{ð}{4}\right)$

   [Ans. $\frac{e}{\sqrt{2}}\,\{1 + (x-1) - \left(y - \frac{ð}{4}\right) + \frac{(x-1)^2}{2} - (x-1)\left(y - \frac{ð}{4}\right) - \frac{1}{2}\left(y - \frac{ð}{4}\right)$

6. Expand $(1+x+y)^{2\ 1/2}$ at (1, 0)

7. Expand $\sin(x+h)(y+k)$ by Taylor's Theorem

   Ans. $\sin xy + (hy + kx)\cos xy + hk \cos xy - \frac{1}{2}(hy + kx)^2 \sin xy + ....\}$]

8. Obtain the expansion of $\tan^{-1}\frac{y}{x}$ about (1, 1) up to the third degree terms.

Ans. $\frac{ð}{4} - \frac{1}{2}(x-1) + \frac{1}{2}(y-1) + \frac{1}{4}(x-1)^2 - \frac{1}{4}(y-1)^2$

$- \frac{1}{12}(x-1)^3 - \frac{1}{4}(x-1)^2(y-1) + \frac{1}{4}(x-1)(y-1)^2 + \frac{1}{12}(y-1)^3 + ....]$

9. Expand $f(x, y) = y^x$ in the neighbourhood of (1, 1) up to the terms of second degree.

Ans. $1+(y-1) + (x-1)(y-1) + .........$]

10. Expand $(x^2y+\sin y+e^x)$ in powers of $(x-1)$ and $(y-ð)$ through quadratic terms by using Taylor's series.

Ans. $(ð + e) + (2ð+e)(x-1) + \frac{1}{2}(x-1)^2 (2ð + e) + 2(x-1)(y-ð) + ..................$ ]

## 4.1. MAXIMA AND MINIMA OF FUNCTIONS OF TWO VARIABLES

A function f(x, y) is said to have a maximum value at x=a, y=b if *f*(a, b) > f (a+h, b+k) for small values of h and k, positive or negative.

A function f(x, y) is said to have a minimum value at x=a, y=b if *f*(a, b) < *f* (a+h, b+k) for small values of h and k, positive or negative.

A maximum or a minimum value of a function is called an extreme value.

## 4.2. RULE TO FIND THE EXTREME VALUES OF A FUNCTION

z = *f* (x, y)

(i) Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$

(ii) Solve $\frac{\partial z}{\partial x} = 0$ and $\frac{\partial z}{\partial y} = 0$ and simultaneously.

Let $(x_1, y_1); (x_2, y_2);$ .......... be the solutions of these equations.

(iii) For each solution in step (ii), find

$$r = \frac{\partial^2 f}{\partial x^2},\quad s = \frac{\partial^2 f}{\partial x \partial y},\quad t = \frac{\partial^2 f}{\partial y^2}$$

(iv) (a) If $rt - s^2 > 0$ and $r < 0$ for a particular solution (x1, y1) of step (ii), then z has a maximum value at $(x_1, y_1)$

(b) If $rt - s^2 > 0$ and $r > 0$ for a particular solution (x1, y1) of step (ii), then z has a minimum value at $(x_1, y_1)$

(c) If $rt - s^2 < 0$ for a particular solution $(x_1, y_1)$ of step (ii), then z has no extreme value at $(x_1, y_1)$

(d) If $rt - s^2 = 0$, the case is doubtful and requires further investigation.

Note: The points $(x_1, y_1)$; $(x_2, y_2)$; ..................... are called stationary points and the values of f (x, y) at these points are called stationary values.

### ILLUSTRATIVE EXAMPLES

**Example 1.** Examine the function $x^3 + y^3 - 3axy$ for maxima and minima.

Sol. Here $f(x, y) = x^3 + y^3 - 3axy$

$f_x = 3x^2 - 3ay \qquad f_y = 3y^2 - 3ax$

$r = f_{xx} = 6x, \quad s = f_{xy} = -3a, \qquad t = f_{yy} = 6y$

Now $f_x = 0$ and $f_y = 0$

$\Rightarrow \quad x^2 - ay = 0$

$y^2 - ax = 0$

From (1), $y = \frac{x^2}{a}$

$\therefore$ From (2), $\frac{x^4}{a^2}$ - ax = 0 or $x(x^3-a^3) = 0$

or $\quad x = 0, a$

When x = 0, y = 0, when x = a, y = a

$\therefore$ There are two stationary points (0, 0) and (a, a).

Now $\quad rt - s^2 = 36xy - 9a^2$

At (0, 0), $\quad rt - s^2 = -9a^2 < 0$

⇨ There is no extreme value at (0, 0)

At (a, a), $rt - s^2 = 36a^2 - 9a^2 = 27a^2 > 0$

⇨ *f*(x, y) has extreme value at (a, a)

Now $\quad r = 6a$

If a > 0, r >0 so that *f* (x, y) has a minimum value at (a, a)

Minimum value = *f* (a, a) = $a^3+a^3-3a^3= -a^3$

If a<0, r <0 so that *f*(x, y) has a maximum value at (a, a)

Maximum value = *f* (a, a) = $a^3+a^3-3a^3= -a^3$

## HW

1. Examine for extreme values:

(i) $x^2+y^2+3x+12$

(ii) $x^3+y^3+3xy$

(iii) $3x^2-y^2+x^3$

(iv) $x^2y^2-5x^2-8xy-5y^2$

(v) $x^3+3xy^2-15x^2-51y^2+72x$

Ans: (i) Min. value = 3at (-3, 0) (ii) Max value = 1 at (-1, -1)

(iii) Max. value = 4at (-2, 0) (iv) Max value = 0 at (0, 0)

(v) Max. value = 112at (4, 0) (ii) Min. value = 108 at (6, 0)

2. Find the stationary points of

$f(x, y) = y^2+4xy+3x^2+x^3$

Examine them for the extreme values of the function.

Ans. (0, 0), $\left(\frac{2}{3}, \frac{-4}{3}\right)$; Main value = $-\frac{4}{27}$ at $\left(\frac{2}{3}, \frac{-4}{3}\right)$

3. (i) Investigate $f(x) = x^5 - 5x^4$ for maxima and minima.

Ans. Min. value = - 256 at x = 4

(ii) Find the minimum value of the function

$f(x, y) = x^2+y^2+xy+ax+by$

where a and b arae constants

Ans. $-\frac{1}{9}(3a^2+3b^2-2ab)$

4. When travelling x km per hour a truck burns diesel oil att eh rate of $\frac{1}{300}\left(\frac{900}{x}+x\right)$

litres per km. If diesel oil costs 40P per litre and the driver is paid Rs. 1.50per hour, find the steady speed that will minimise the total cost of a trip of 500km.

Hint. Total cost of trip is C = $\frac{500}{300}\left(\frac{900}{x}+x\right) \times 0.40 + \frac{900}{x} \times 1.50$

Ans. 45km/hour

5. Determine the points where the function

$x^2y+xy^2-axy$

has a maximum or a minimum

Ans. Min. value = $-\frac{a^3}{27}$ at $\left(\frac{a}{3}, \frac{a}{3}\right)$ when a > 0

Max. value = $\frac{a^3}{27}$ at $\left(\frac{a}{3}, \frac{a}{3}\right)$ when a < 0

6. Find the stationary points of the function $z = x^3y^3 (12-x-y)$ satisfying the condition $x>0, y>0$ and examine their nature.

Ans. Maximum at (6, 4)

7. Find the minimum value of $x^2+y^2+z^2$ when

(i) $x+y+z = 3a$ (ii) $xyz = a^3$

Ans. (i) $3a^2$ (ii) $3a^2$

8. Find the points of the surface $z = xy+1$ nearest to the origin

Ans. $(0, 0, \pm 1)$

9. Which point of the sphere $x^2+y^2+z^2 = 1$ is farthest from the point (2, 1, 3)

Ans. $\left(\frac{-2}{\sqrt{14}}, \frac{-1}{\sqrt{14}}, \frac{-3}{\sqrt{14}}\right)$

10. Given $f(x, y, z) = \frac{5xyz}{x+2y+4z}$ find the values of x, y, z for which $f(x, y, z)$ is a maximum,

subject to the condition $xyz = 8$

Ans. (4, 2, 1)

11. A rectangular box, open at the top is to have a volume of 32 c.c. Find the dimensions of the box requiing least material for its construction.

Ans. 4cmx4cmx2cm

12. Find the dimensionn of the rectangular box without a top, of maximim capacity whose surface is 108sq cm.

Ans. Length = Breadth = 6m, Height = 3cm

13. The sum of three numbers is constant. Prove that their product is a maximum when they are equal.

14. Find the point on the plane $ax+by+cz = p$ at which the functions $\phi = x^2+y^2+z^2$ has a minimum value and find this minimum $\phi$

Ans. $\left(\frac{ap}{\Sigma a^2}, \frac{bp}{\Sigma a^2}, \frac{cp}{\Sigma a^2}; \frac{p^2}{\Sigma a^2}\right)$

15. If x increases at the rate of 2cm/sec at the instant when x = 3 and y = 1, at what rate must y be changing in order that the function $2xy - 3x^2y$ shall be neither increasing nor decreasing?

Ans. Decreasing at $\frac{32}{21}$ cm/sec

Hint. Let $f(x, y) = 2xy-3x^2y$, then

$$\frac{df}{dt} = \frac{\partial f}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial f}{\partial f} \cdot \frac{dy}{dt}$$

Put x = 3, y = 1, $\frac{dx}{dt} = + 2$ cm/sec

$f$ is neither increasing nor decreasing when $\frac{df}{dt} = 0$