MODULE III

FINITE DIFFERENCES

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Finite differences : Suppose that the function $y = f(x)$ is tabulated for the equally spaced values $x = x_0, x_0+h, x_0+2h \ldots\ldots x_0+nh$ giving $y = y_0, y_1, y_2, \ldots\ldots, y_n$. To determine the values of $f(x)$ or $f'(x)$ for some intermediate values of x,the following 3 types of differences are very useful.

Forward differences : The differences $y_1 - y_0,\ y_2-y_1, \ldots. y_n-y_{n-1}$ are called the first forward differences and is denoted by $\Delta y_0, \Delta y_1, \Delta y_2 \ldots\ldots\ldots\ldots \Delta y_{n-1}$ where Δ is the forward difference operator. In general $\Delta y_n = y_{n+1} - y_n$

Second forward differences are $\Delta^2 y_n = \Delta y_{n+1} - \Delta y_n$

In general $\Delta^p y_n = \Delta^{p-1} y_{n+1} - \Delta^{p-1} y_n$ is the p^{th} forward differences.

Forward difference table

Value of x	Value of y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
X_0	y_0					
		Δy_0				
	y_1		$\Delta^2 y_0$			
x_{0+h}		Δy_1		$\Delta^3 y_0$		
	y_2		$\Delta^2 y_1$		$\Delta^4 y_0$	
		Δy_2		$\Delta^3 y_1$		$\Delta^5 y_0$
x_{0+2h}	y_3		$\Delta^2 y_2$		$\Delta^4 y_1$	
		Δy_3		$\Delta^3 y_2$		
	y_4		$\Delta^2 y_3$			
x_{0+3h}		Δy_4				
	y_5					
x_{0+4h}						
x_{0+5h}						

In a difference table , x is called the argument and y the function or the entry.y0 , the first entry is called the leading term and element $\Delta y_0,\ \Delta^2 y_0, \Delta^3 y_0 \ldots\ldots\ldots$ are called the leading differences.

2) Backward differences : The differences $y_1 - y_0,\ y_2-y_1, \ldots. y_n-y_{n-1}$ are called the first backward differences and is denoted by $\nabla y_1,\ \nabla y_2,\ \nabla y_3 \ldots\ldots \nabla y_n$ where ∇y is the backward difference operator. In general ? $y_n = y_n - y_{n-1}$

Second forward differences are $\nabla^2 y_n = \nabla y_n - \nabla y_{n-1}$

Backward difference table

Value of x	Value of y	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$	$\nabla^5 y$
X_0	y_0					
		∇y_1				
	y_1		$\nabla^2 y_2$			
x_{0+h}		∇y_2		$\nabla^3 y_3$		
	y_2		$\nabla^2 y_3$		$\nabla^4 y_4$	
		∇y_3		$\nabla^3 y_4$		$\nabla^5 y_5$
x_{0+2h}	y_3		$\nabla^2 y_4$		$\nabla^4 y_5$	
		∇y_4		$\nabla^3 y_5$		
	y_4		$\nabla^2 y_5$			
x_{0+3h}		∇y_5				
	y_5					
x_{0+4h}						
x_{0+5h}						

Central differences : The central difference operator δ is defined by $y_1 - y_0 = \delta y_{1/2}$.
$y_2 - y_1 = \delta y_{3/2}, \ldots . y_n - y_{n-1} = \delta y_{n-1/2}$.

Similarly $\delta y_{3/2} - \delta y_{1/2} = \delta^2 y_1$, $\boldsymbol{\delta y_{5/2} - \delta_{3/2}} = \delta^2 y_2$ ....

Central difference table

Value of x	Value of y	1 st difference	2 nd difference	3rd difference
X_0	y_0			
		$\delta y_{1/2}$		
	y_1		$\delta^2 y_1$	
x_{0+h}		$\delta y_{3/2}$		$\delta^3 y_{3/2}$
	y_2		$\delta^2 y_2$	
		$\boldsymbol{\delta y_{5/2}}$		
x_{0+2h}	y_3			
x_{0+3h}				

Shift Operator(E)

The Shift operator is defined as

$$E\,y_k = y_{k+1}$$

$$E\,f(x) = f(x+h)$$

$$E^2 f(x) = f(x+2h) \ldots\ldots\ldots\ldots E^n f(x) = f(x+nh)$$

Averaging Operator (μ)

The averaging operator is denoted by μ and is defined as

$$\mu\, f(x) = \frac{f(x+h/2) + f(x-h/2)}{2} = \frac{E^{1/2} + E^{-1/2}}{2} f(x)$$

Relation between the operators

1) Show that $\Delta = E - 1$

$\Delta f(x) = f(x+h) - f(x)$

$= E f(x) - f(x)$

$= (E-1)f(x)$

ie, $\Delta = E - 1$

Additional questions

1)S.T $\nabla = 1 - E^{-1}$

2)S.T $\delta = E^{1/2} - E^{-1/2}$

3)S.T $\mu = \frac{E^{1/2} + E^{-1/2}}{2}$

4)S.T $\Delta = E\nabla = \nabla E = \delta E^{1/2}$

5)S.T $\delta E^{1/2} = \nabla$

Problems

1)P.T a) $hD = \log(1 + \Delta) = -\log(1-\nabla) = \sinh^{-1}(\mu\delta)$

b) $(E^{1/2} - E^{-1/2})(1+\Delta)^{1/2} = 2 + \Delta$

c) $\Delta^3 y_2 = \nabla^3 y_5$

DIFFERENCES OF A POLYNOMIAL

The n^{th} difference of a polynomial of the n^{th} degree are constant and all the higher order differences are zero.

Factorial Notation : A product of the form x(x-1)(x-2)......(x-r+1)is denoted by $[x]^r$ and is called a factorial.

Eg: $[x] = x$

$[x]^2 = x(x-1)$

....

$[x]^n = x(x-1)(x-2)........(x-n+1)$

Interpolation

Interpolation is the technique of estimating the value of a function for any intermediate value of the independent variable.

NEWTON'S FORWARD INTERPOLATION FORMULA

$$y_p = y_0 + p\,\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \dots\dots + \frac{p(p-1)(p-2)\dots\dots(p-n+1)}{n!}\Delta^n y_0$$

Proof

Let the function $y = f(x)$ takes the values $y_0, y_1, y_2, \dots\dots, y_n \dots\dots$ corresponding to the values $x_0, x_0 + h, x_0 + 2h \dots\dots x_0 + nh$ Suppose it is required to evaluate $f(x)$ for $x = x_0 + ph$ where p is any real number.

For any real number p we can define E such that $E^p f(x) = f(x+ph)$

ie, $y_p = f(x_0 + ph) = E^p f(x_0)$

$= (1 + \Delta)^p \Delta f(x_0) = (1 + \Delta)^p \Delta y_0$

$= \left(1 + p\,\Delta + \frac{p(p-1)}{2!}\Delta^2 + \frac{p(p-1)(p-2)}{3!}\Delta^3 + \dots\dots\right) y_0$

$= y_0 + p\,\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \dots\dots \quad (1)$

If $y = f(x)$ is a polynomial of nth degree , then $\Delta^{n+1} y_0$ and higher order differences will be zero .Hence (1) will become,

$$y_p = y_0 + p\,\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \dots\dots + \frac{p(p-1)(p-2)\dots\dots(p-n+1)}{n!}\Delta^n y_0$$

NEWTON'S BACKWARD INTERPOLATION FORMULA

$$y_p = y_0 + p \blacktriangledown y_0 + \frac{p(p-1)}{2!} \blacktriangledown^2 y_0 + \frac{p(p-1)(p-2)}{3!} \blacktriangledown^3 y_0 + \dots\dots + \frac{p(p-1)(p-2)\dots\dots(p-n+1)}{n!} \blacktriangledown^n y_0$$

Proof

Let the function $y = f(x)$ takes the values $y_0, y_1, y_2, \dots\dots, y_n \dots\dots$ corresponding to the values $x_0, x_0 + h, x_0 + 2h \dots\dots x_0 + nh$ Suppose it is required to evaluate $f(x)$ for $x = x_0 + ph$ where p is any real number.

For any real number p we can define E such that $E^p f(x) = f(x+ph)$

ie, $y_p = f(x_n + ph) = E^p f(x_n) = (E^{-1})^{-p} f(x_n)$

$= (1 - \blacktriangledown)^{-p} y_n$

$= \left(1 + p \blacktriangledown + \frac{p(p-1)}{2!} \blacktriangledown^2 + \frac{p(p-1)(p-2)}{3!} \blacktriangledown^3 + \dots\dots\right) y_n$

$$= y_n + p \nabla y_n + \frac{p(p-1)}{2!} \nabla^2 y_n + \frac{p(p-1)(p-2)}{3!} \nabla^3 y_n + \ldots\ldots \quad (1)$$

If $y = f(x)$ is a polynomial of nth degree , then $\nabla^{n+1} y_n$ and higher order differences will be zero .Hence (1) will become,

$$y_p = \mathbf{y_n} + p \nabla y_n + \frac{p(p-1)}{2!} \nabla^2 y_n + \frac{p(p-1)(p-2)}{3!} \nabla^3 y_n + \ldots\ldots + \frac{p(p-1)(p-2)\ldots\ldots(p-n+1)}{n!} \nabla^n y_n$$

Problem

1)Find the value of y at the points x = 11 and x = 55 from the following data by using Newton's interpolation formula

x :	10	20	30	40	50	60
f:	5	26	47	68	89	110

x	y	Δy	$\Delta^2 y$
10	5		
		21	
20	26		0
		21	
30	47		0
		21	
40	68		0
		21	
50	89		0
		21	
60	110		

Here $x_0 = 10$, $p = \frac{x - x_0}{h} = 0.1$

$$y_{11} = \mathbf{y_0} + p \Delta y_0 + \frac{p(p-1)}{2!} \Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!} \Delta^3 y_0 + \ldots\ldots + \frac{p(p-1)(p-2)\ldots\ldots(p-n+1)}{n!} \Delta^n y_0$$

$$= 5 + 0.1 \times 21 + 0 = 7.1$$

$$y_{55} = \mathbf{y_n} + p \nabla y_n + \frac{p(p-1)}{2!} \nabla^2 y_n + \frac{p(p-1)(p-2)}{3!} \nabla^3 y_n + \ldots\ldots + \frac{p(p-1)(p-2)\ldots\ldots(p-n+1)}{n!} \nabla^n y_n$$

$$= 110 + -0.5 \times 21 = 99.5$$

Additional Questions

1)Find the cubic polynomial which takes the following values

x : 0 1 2 3

f : 1 2 1 10 .Hence or otherwise evaluate f(4)

2) Given sin45°= 0.7071, sin 50° = 0.7660,sin 55°=0.8192, sin 60°= 0.8660, find sin 52°using Newton's forward interpolation formula.

3)Find a polynomial of degree 3 which takes the following values

x :	3	4	5	6
f :	6	24	60	120

4)From the following data , find f at x = 43 and x = 84

x :	40	50	60	70	80	90
f :	184	204	226	250	276	304

CENTRAL DIFFERENCE FORMULA

STIRLING'S FORMULA:

$$y_p = y_0 + p\left\{\frac{\Delta y_0 + \Delta y_{-1}}{1\ \ 2}\right\} + \frac{p^2}{2!}\Delta^2 y_{-1} + \frac{p(p^2-1)}{3!}\left\{\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right\} + \ldots\ldots\ldots\ldots$$

where $p = \frac{x - x_0}{h}$, h is the length of the interval.

Problem

1) Generate the central difference table from the following data and evaluate the value of y(11).

x :	2	6	10	14	18
y:	21.857	21.025	20.132	19.145	18.057

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
2	21.857				
		-0.832			
6	21.025		-0.001		
		-0.833		-0.153	
10	20.132		-0.154		0.206
		-0.987		-0.053	
14	19.145		-0.101		
		-1.088			
18	18.057				

By stirling's formula,

$$y_p = y_0 + p\left\{\frac{\Delta y_0 + \Delta y_{-1}}{2}\right\} + \frac{p^2}{2!}\Delta^2 y_{-1} + \frac{p(p^2-1)}{3!}\left\{\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right\} + \ldots\ldots\ldots\ldots$$

$$p = \frac{x - x_0}{h} = 1/4$$

$$y_{11} = 20.132 + 1/4\{\frac{-0.833+-0.987}{2}\} + \frac{1/16}{2!}\times -0.154 + \frac{1/4(1/16-1)}{3!}\frac{(-0.153+0.053)}{2} +$$

$$\frac{\{1/16(1/16-1)\}}{4!}\times 0.206 + ... = 19.9011377$$

Additional problems

1 Find y(35) by using the Stirling's formula from the following data

x :	20	30	40	50
y :	512	439	346	243

2 Use Stirling's formula to evaluate f(1.23) from the following data

x :	1	1.1	1.2	1.3	1.4
f(x) :	0.841	0.891	0.932	0.963	0.985

LAGRANGE'S FORMULA FOR UNEQUAL INTERVALS

If $y = f(x)$ takes the values $y_0, y_1, y_2....y_n$ corresponding to $x = x_0, x_1, x_2.......x_n$ then $y = f(x) = A_0y_0 + A_1y_1 + ...+ A_ny_n$ where

$$A_0 = \frac{(x-x_1)(x-x_2).... (x-x_n)}{(x_0-x_1)(x_0-x_2)...(x_0-x_n)}$$

$$A_1 = \frac{(x-x_0)(x-x_2).... (x-x_n)}{(x_1-x_0)(x_1-x_2)...(x_1-x_n)}$$

....

$$A_n = \frac{(x-x_0)(x-x_1).... (x-x_{n-1})}{(x_n-x_0)(x_n-x_1)...(x_n-x_{n-1})}$$

Proof: Let $y = f(x)$ be the function which takes the values $(x_0,x_1), (x_1,x_2)....(x_n,y_n)$ .Since there are (n+1) pairs of values of x&y we can represent f(x) by a polynomial in x of degree n and the polynomial is given by

$$y = f(x) = a_0(x-x_1)(x-x_2)...(x-x_n)+a_1(x-x_2)(x-x_3)...(x-x_n)+...+a_n(x-x_0)(x-x_1)...(x-x_{n-1}).......(1)$$

Putting $x = x_0$ & $y=y_0$ in (1) we get

$y_0 = a_0(x_0-x_1)(x_0-x_2)...(x_0-x_n)$

$$a_0 = \frac{y_0}{(x_0-x_1)(x_0-x_2)...(x_0-x_n)}$$

Putting $x = x_1$, $y = y_1$ in (1) we get

$y_1 = a_1(x_1-x_2)(x_1-x_3)...(x_1-x_n)$

$$a_1 = \frac{y_1}{(x_1-x_2)(x_1-x_3)...(x_1-x_n)}$$

Proceeding the same way , we find $a_2,a_3 ...a_n$ substituting the values of $a_0,a_1,a_2...a_n$ in (1) we get

$$y = \frac{(x-x_1)(x-x_2)....(x-x_n)}{(x_0-x_1)(x_0-x_2)...(x_0-x_n)} y_0 + \frac{(x-x_0)(x-x_2)....(x-x_n)}{(x_1-x_0)(x_1-x_2)...(x_1-x_n)} y_1 + ...+ \frac{(x-x_0)(x-x_1)....(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)...(x_n-x_{n-1})} y_n$$

$$= A_0y_0 + A_1y_1+.....+A_ny_n$$

Problem

1 Using Lagrange's interpolation formula to find f(8) from the following table

x :	1	4	7	9	15
y :	8	11	13	14	16

By Lagrange's interpolation formula

$$= \frac{(x-x_1)(x-x_2)....(x-x_n)}{(x_0-x_1)(x_0-x_2)...(x_0-x_n)} y_0 + \frac{(x-x_0)(x-x_2)....(x-x_n)}{(x_1-x_0)(x_1-x_2)...(x_1-x_n)} y_1 + ...+ \frac{(x-x_0)(x-x_1)....(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)...(x_n-x_{n-1})} y_n$$

here $x = 8$, $x_0 = 1, x_1 = 4$, $x_2=7$, $x_3 =9, x_4 = 15$

$y_0= 8, y_1=11, y_2= 13, y_3=14, y4=16$

$y = 2.092171832$

Additional problems

1 Given the following table, find the value of the polynomial at x = 0.9

x :	0	1	2	4
f(x) :	5	14	41	98

2 Using Lagrange's formula , find the interpolating polynomial , given that y(1) = -3

y(3) = 9,y(4)=30 & y(6) = 132

NEWTON'S DIVIDED DIFFERENCE FORMULA FOR UNEQUAL INTERVALS

Divided difference : Let $y_0,y_1,y_2.....y_n$ be the values of the function y = f(x) corresponding to the arguments $x_0,x_1,x_2 ....x_n$.

The first divided difference of f(x) for the arguments x_0&x_1 is denoted by $f(x_0,x_1)$ or

$[x_0\ x_1]$ and is defined as $[x_0\ x_1] = \frac{y_1 - y_0}{x_1 - x_0}$

The second divided difference of f(x) for the arguments x_0 , x_1,x_2 is denoted by $f(x_0,x_1,x_2)$ or $[x_0\ x_1\ x_2]$ and is defined as $[x_0\ x_1\ x_2] = \frac{[x_1\ x_2] - [x_0\ x_1]}{[x_2\ x_0]}$

The nth divided difference of f(x) for the arguments $x_0,x_1,x_2 ....x_n$ is denoted by $[x_0,x_1,x_2 ....x_n]$ and is defined as

$$[x_0\ x_1\ x_2 ....x_n] = \frac{[x_0\ x_1\ x_2 ...x_n]- [x_0\ x_1\ x_2 ...x_{n-1}]}{X_n - x_0}$$

NEWTON'S DIVIDED DIFFERENCE FORMULA

Let $y_0,y_1,y_2 ...y_n$ be the values of the function y = f(x) corresponding to the arguments $x_0,x_1,x_2 ....x_n$ then $y = f(x)= y_0 + (x - x_0)[x_0\ x_1] + (x-x_0)(x-x_1) [x_0\ x_1\ x_2] +..$
$+(x-x_0) (x-x_1) (x - x_2)....(x-x_{n-1})[x_0\ x_1\ x_2....x_n]$

Proof : From the definition of divided difference we have

$$[x\ x_0] = \frac{y - y_0}{x - x_0} \quad \text{so that } y = y_0 + (x-x_0) [x\ x_0]......(1)$$

$$[x\ x_0\ x_1] = \frac{[x\ x_0] - [x_0\ x_1]}{x-x_1}$$

ie, $[x\ x_0] = [x_0\ x_1] + [x\ x_0\ x_1] (x-x_1)$

sub. this value of [x x0] in (1) we get

$$y = y_0+(x - x_0)[x_0\ x_1] + (x-x_0) (x-x_1) [x\ x_0\ x_1] \ ....(2)$$

Also $[x\ x_0\ x_1\ x_2] = \frac{[x\ x_0\ x_1] - [x_0\ x_1\ x_2]}{x- x_2}$

which gives $[x\ x_0\ x_1] = [x_0\ x_1\ x_2] + (x-x_2) [x\ x_0\ x_1\ x_2]$

sub. this value in (2) we get

$$\mathbf{y = y_0 +(x-x_0) [x_0\ x_1] +(x-x_0)(x-x_1) [x_0\ x_1\ x_2] +(x-x_0) (x-x_1)(x-x_2) [x_0\ x_1\ x_2\ x_3]}$$

proceeding like this we get

$y = f(x)= y_0 + (x - x_0)[x_0\ x_1] + (x-x_0)(x-x_1) [x_0\ x_1\ x_2] +.....$
$+(x-x_0) (x-x_1) (x - x_2)....(x-xn)[x_0\ x_1\ x_2....x_n]$ .....(a)

If f(x) is a polynomial of degree n, then [x x0 x1…xn] = 0

ie, $y = f(x) = y_0 + (x - x_0)[x_0\, x_1] + (x-x_0)(x-x_1)[x_0\, x_1\, x_2] + \ldots$
$+ (x-x_0)(x-x_1)(x - x_2)\ldots(x-x_{n-1})[x_0\, x_1\, x_2 \ldots x_n]$

Problem

1) Find f(9) by using Newton's divided difference formula from the following table

x : 5 7 11 13 17
f : 150 392 1452 2366 5202

Newton's divided difference table is

x	y	1st difference	2nd difference	3rd difference	4th difference
5	150				
		121			
7	392		24		
		265		1	
11	1452		32		0
		457		1	
13	2366		42		
		709			
17	5202				

Taking x = 9 in the Newton's divided difference formula,

f(9) = 810

Home work

1) Find f(8), f(2),f(15) by using Newton's divided difference formula from the following table.

x : 4 5 7 10 11 13
y: 48 100 294 900 1210 2028

2)Estimate the values of f(22) & f(44) from

x: 20 25 30 35 40 45
y: 354 332 291 260 231 204

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