MODULE IV

DIFFERENCE CALCULUS

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NEWTON'S FORWARD FORMULA AND BACKWARD FORMULA FOR NUMERICAL DIFFERENTIATION

Newton's forward interpolation formula is

$$\mathbf{y = y_0 + p}\,\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \ldots\ldots.$$

Differentiating both sides w.r. to p we have,

$$\frac{dy}{dp} = \Delta y_0 + \frac{2p-1}{2!}\Delta^2 y_0 + \frac{3p^2-6p+2}{3!}\Delta^3 y_0 + \ldots..$$

But $p = \frac{x - x_0}{h}$, $\frac{dp}{dx} = 1/h$

Now $\frac{dy}{dx} = \frac{dy}{dp}\frac{dp}{dx} = 1/h\,[\,\Delta y_0 + \frac{2p-1}{2!}\Delta^2 y_0 + \frac{3p^2-6p+2}{3!}\Delta^3 y_0 + \ldots..]$ ......(1)

At $x = x_0$, $p = 0$ .Hence putting $p = 0$,

$$\frac{dy}{dx} \text{ at } x_0 = 1/h\,[\Delta y_0 - 1/2\,\Delta^2 y_0 + 1/3\,\Delta^3 y_0 - ¼\,\Delta^4 y_0 + \ldots..]$$

Again differtiating (1) w.r.to x we get,

$$\frac{d^2y}{dx^2} = \frac{d}{dp}\left(\frac{dy}{dx}\right)\frac{dp}{dx}$$

$$= 1/h\,[2/2!\,\Delta^2 y_0 + \frac{6p-3}{3!}\Delta^3 y_0 + \frac{12p^2-36p+22}{4!}\Delta^4 y_0 + \ldots]\,1/h$$

Putting p = 0 we obtain

$$\frac{d^2y}{dx^2} \text{ at } x_0 = 1/h^2\,[\,\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}\Delta^4 y_0 - \frac{5}{6}\Delta^5 y_0 + \frac{137}{180}\Delta^6 y_0 + \ldots\ldots..].....(1)$$

$$\frac{d^3y}{dx^3} = 1/h^3\,[\,\Delta^3 y_0 - 2/3\,\Delta^4 y_0 + \ldots\ldots.\ (4)$$

ii) Newton's backward interpolation formula is

$$\mathbf{y = y_n + p}\,\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_n + \ldots\ldots.$$

Differentiating both sides w.r. to p we h

$$\frac{dy}{dx} = \nabla\,\mathbf{y_n + \frac{2p+1}{2}}\,\nabla^2 y_n + \frac{3p2+6p+2}{3!}\nabla^2 y_n + \ldots.$$

since $p = \frac{x - x_n}{h}$, $\frac{dp}{dx} = 1/h$

Now, $\frac{dy}{dx} = \frac{dy}{dp}\frac{dp}{dx} = 1/h\,[\,\nabla y_n + \frac{2p+1}{2!}\nabla^2 y_n + \frac{3p^2+6p+2}{3!}\nabla^3 y_0 + \ldots..]$ ......(5)

At $x = x_n$, $p = 0$ .Hence putting $p = 0$ we get,

$\frac{dy}{dx}$ at x_n = 1 / h [$\nabla y_n + 1/2 \nabla^2 y_n + 1/3 \nabla^3 y_n$ .....] .....(6)

Again differentiating (5) w.r. to x we get,

$$\frac{d^2y}{dx^2} = \frac{d}{dp}\left(\frac{dy}{dx}\right)\frac{dp}{dx}$$

$$= 1/h \left[\nabla^2 y_n + \frac{6p+6}{3!} \nabla^3 y_n + \frac{6p^2+18p+11}{12} \nabla^4 y_n + ...\right]$$

Putting p = 0 we obtain

$$\frac{d^2y}{dx^2} \text{ at } x_n = 1/h^2 \left[\nabla^2 y_n + \nabla^3 y_n + \frac{11}{12} \nabla^4 y_n + \frac{5}{6} \nabla^5 y_n \right] ....(7)$$

$$\frac{d^3y}{dx^3} \text{ at } x_n = 1/h^3 \left[\nabla^3 y_n + \frac{3}{2} \nabla^4 y_n \right] ....(8)$$

<u>Problems</u>

1 Given that x : 1 1.1 1.2 1.3 1.4 1.5 1.6

y : 7.989 8.403 8.781 9.129 9.451 9.750 10.031

find $\frac{dy}{dx}$ & $\frac{d^2y}{dx^2}$ at x = 1.1 & 1.6

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$	$\Delta^6 y$
1.0	7.989						
		0.414					
1.1	8.403		- 0.036				
		0.378		0.006			
1.2	8.781		- 0.030		-0.002		
		0.348		0.004		0.002	
1.3	9.129		-0.026		0.000		-0.003
		0.322		0.004		-0.001	
1.4	9.451		-0.023		-0.001		
		0.299		0.005			
1.5	9.750		-0.018				
		0.281					
1.6	10.031						

We have $\frac{dy}{dx}$ at x_0 = 1 / h [$\Delta y_0 - 1/2 \Delta^2 y_0 + 1/3 \Delta^3 y_0 - ¼ \Delta^4 y_0 +$ .....]

$\frac{dy}{dx}$ at 1.1 = 3.946

$$\frac{d^2y}{dx^2} \text{ at } x_0 = 1/h^2 \left[\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12} \Delta^4 y_0 - \frac{5}{6} \Delta^5 y_0 + \frac{137}{180} \Delta^6 y_0 + ...\right] .$$

$\frac{d^2y}{dx^2}$ at 1.1 = -3.545

we use the above difference table and the backward difference operator ?

$\frac{dy}{dx}$ at x_n $= 1 / h\, [\nabla y_n + 1/2\ \nabla^2 y_n + 1/3\ \nabla^3 y_n + \ldots\ldots]$

$\frac{dy}{dx}$ at 1.6 $= 2.727$

$\frac{d^2y}{dx^2}$ at $x = 1.6$ $= -1.703$

1) Find the first and second derivatives of the function tabulated below at the points x = 2 and x = 1.9

x:	1	1.2	1.4	1.6	1.8	2
y:	0	0.128	0.544	1.296	2.432	4

2) Find the first and second derivatives of the function tabulated below at the points x =1.1 and x = 1.0

x:	1	1.2	1.4	1.6	1.8	2
y:	0	0.128	0.544	1.296	2.432	4

3) Find the first two derivatives of $(x)^{1/3}$ at x = 50 and x = 56 given the table below

x :	50	51	52	53	54	55	56
$y = x^{1/3}$:	3.684	3.7084	3.7325	3.7563	3.7798	3.8030	3.8259

DERIVATIVES USING STIRLING ' S FORMULA

$$\frac{dy}{dx} \text{ at } x = x_0 = \frac{1}{h}\left\{ \frac{1}{2}(\Delta y_0 + \Delta y_{-1}) - \frac{1}{12}(\Delta^3 y_{-1} + \Delta^3 y_{-2}) + \frac{1}{60}(\Delta^5 y_{-2} + \Delta^5 y_{-3}) \ldots\ldots \right\}$$

$$\frac{d^2y}{dx^2} \text{ at } x = x_0 = \frac{1}{h^2}\left\{ \Delta^2 y_{-1} - \frac{1}{12}\Delta^4 y_{-2} + \ldots \right\}$$

$$\frac{d^3y}{dx^3} \text{ at } x = x_0 = \frac{1}{h^3}\left\{ \tfrac{1}{2}(\Delta^3 y_{-1} - \Delta^3 y_{-2}) + \ldots \right\}$$

Problems

1 Find the first and second derivatives of the function tabulated below at x = 0.6

x :	0.4	0.5	0.6	0.7	0.8
y:	1.5836	1.7974	2.0442	2.3275	2.6511

Since x = 0.6 is in the middle of the table we will use stirling's formula

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
0.4	1.5836				
		0.2138			
0.5	1.7974		0.0330		
		0.2468		0.0035	
0.6	2.0442		0.0365		0.0003
		0.2833		0.0038	
0.7	2.3275		0.0403		
		0.3236			
0.8	2.6511				

By Stirling's formula

$$\frac{dy}{dx} \text{ at } x = x_0 = \frac{1}{h}\left\{ \frac{1}{2}(\Delta y_0 + \Delta y_{-1}) - \frac{1}{12}(\Delta^3 y_{-1} + \Delta^3 y_{-2}) + \frac{1}{60}(\Delta^5 y_{-2} + \Delta^5 y_{-3}) \ldots\ldots \right\}$$

$$\frac{d.y}{dx} \text{ at } x = 0.6 = 2.64442$$

$$\frac{d^2 y}{d x^2} \text{ at } x = x_0 = \frac{1}{h^2}\left\{ \Delta^2 y_{-1} - \frac{1}{12}\Delta^4 y_{-2} + \ldots \right\}$$

$$\frac{d^2 y}{d x^2} \text{ at } x = 0.6 = 3.6475$$

Home work

1) Find the value of f ' (0.5) using stirling's formula from the following data

x :	0.35	0.4	0.45	0.5	0.55	0.6	0.65
y :	1.521	1.506	1.488	1.467	1.444	1.418	1.389

NUMERICAL INTEGRATION

1)NEWTON – COTES QUADRATURE FORMULA

$$I = nh\left[y_0 + \frac{n}{2}\Delta y_0 + \frac{n(2n-3)}{12}\Delta^2 y_0 + \ldots\ldots\ldots \right]$$

2) TRAPEZOIDAL RULE

$$\int_{x_0}^{x_0+ph} f(x)\,dx = \frac{h}{2}\left[(y_0 + y_n) + 2(y_1 + y_2 + \ldots\ldots + y_{n-1}) \right]$$

3) SIMPSON'S ONE-THIRD RULE

$$\int_{x_0}^{x_0+ph} f(x)\,dx = \frac{h}{3}\left[(y_0+y_n) + 4(y_1+y_3+......+y_{n-1}) + 2(y_2+y_4+......+y_{n-2}) \right]$$

4)SIMPSON'S THREE-EIGTH RULE

$$\int_{x_0}^{x_n} f(x)\,dx = \frac{3h}{8}\left[(y_0+y_n) + 3(y_1+y_2+y_3.....) + 2(y_3+y_6+......) \right]$$

1 Evaluate $\int_0^6 \frac{dx}{1+x^2}$ by using i) Trapezoidal rule

ii) Simpson's 1/3 rule

iii) Simpson's 3/8 rule

Divide the integral (0,6) into six parts each of width h = 1

x :	0	1	2	3	4	5	6
y :	1	0.5	0.2	0.1	0.0588	0.0385	0.072

By Trapezoidal rule

$$\int_{x_0}^{x_0+ph} \frac{1}{1+x^2}\,dx = \frac{h}{2}\left[(y_0+y_6) + 2(y_1+y_2+y_4+y_5) \right] = 1.4108$$

By Simpson's 1/3 rule

$$\int_{x_0}^{x_0+ph} \frac{1}{1+x^2}\,dx = \frac{h}{3}\left[(y_0+y_6) + 4(y_1+y_3+y_5) + 2(y_2+y_4) \right] = 1.3662$$

By Simpson's 3/8 rule

$$\int_{x_0}^{x_0+ph} \frac{1}{1+x^2}\,dx = \frac{3h}{8}\left[(y_0+y_6) +3(y_1+y_2+y_3 + y_4 + y_5) + 2y_3 \right] = 1.3571$$

<u>Home work</u>

1) Use trapezoidal rule to evaluate $\int_0^1 x^3 dx$ considering 5 sub-intervals

2) Evaluate $\int_{-2}^{3} x^2 dx$ by trapezoidal rule

3) Evaluate $\int_{-3}^{3} x^4 dx$ by using trapezoidal rule and Simpson's rule

Difference Equations

An equation which expresses a relation between the independent variable , the dependent variable and the successive differences of the dependent variable is called a difference equation.

Eg:$\Delta^4 y_x + 5\Delta^4 y_x + 6\,\Delta y_x + 18\; y_x = x + \sin x$ is the difference equation . Since Δ is the forward difference and y_x is a function of x.

<u>Order and degree of the difference equation</u>

The order of a difference equation written in the form free from Δ's , is the difference between the highest and lowest subscipts of y or arguments of y. Thus the order of

$y_{x+3} - 5\,y_{x+2} + 7\;y_{x+1} + y_x = 10x$ is $(x+3) - x = 3$

The degree of a difference equation written in the form free from Δ's , is the highest power of the y's.

Eg: $(E^2 - 5E + 16)\,y_x = e^{x}$ is of degree 1

<u>Solution of a difference equation</u>

A solution of a difference equation is a function of its variable which satisfies the difference equation .

<u>General solution of the difference equation</u>

The general solution or complete solution of a difference equation is the sum of two functions.

<u>Linear difference equation</u>

A Linear difference equation with constant coefficients is of the form

$y_{n+r} + a_1\, y_{n+r-1} + a_2\; y_{n+r-2} + \ldots\ldots\ldots\ldots\ldots + a_r\, y_n = f(n)$ where a1,a2............... are constants.

The complete soln of a linear difference equation with constant coefficients is

$y_n = CF + PI$

Where CF is the complementary function and PI is the particular integral.

<u>Steps for finding CF</u>

$y_{n+r} + a_1\, y_{n+r-1} + a_2\; y_{n+r-2} + \ldots\ldots\ldots\ldots\ldots + a_r\, y_n = 0$ where a_1, a_2 ,.....are constants.

i write the equation in the symbolic form

ii ie, $(E^r + a_1\, E^{r-1} + \ldots\ldots + a_r)y_r = 0$

iii Write down the auxiliary equation

ie, $E^r + a_1\, E^{r-1} + \ldots\ldots + a_r = 0$

write down the solution as follows

	Roots for A.E.	Solution (CF)
1.	$\lambda_1,\lambda_2,\lambda_3,\ldots\ldots\lambda_n$ (Real and distinct roots)	$c_1\,\lambda_1^{\,n} + c_2\lambda_2^{\,n} + \ldots + c_n\lambda_n^{\,n}$ ($c_1,c_2,\ldots c_n$ are constants)

2.	$\lambda_1,\lambda_1,\lambda_3,.....\lambda_n$ (2 real and equal roots)	$(c_1 + c_2 n)\lambda_1^n + \lambda_3^n + ...$
3.	$\lambda_1,\lambda_1,\lambda_1,\lambda_4$ .... (3 real and equal roots)	$(c_1 + c_2 n + c_3 n^2)\lambda_1^n + ...$
4.	$\alpha + i\beta$, $\alpha - i\beta$	$r^n(c_1 \cos n\theta + c_2 \sin n\theta)$ where $r = \sqrt{\alpha^2 + \beta^2}$ and $\theta = \tan^{-1}(\beta/\alpha)$

Problems

1) Solve the difference equation $U_{n+3} - 2U_{n+2} - 5U_{n+1} + 6U_n = 0$

Symbolic form of the given equation is

$(E^3 U_n - 2E^2 U_n - 5E U_n + 6U_n) = 0$

$(E^3 - 2E^2 - 5E + 6)U_n = 0$

The auxilliary equation is

$E^3 - 2E^2 - 5E + 6 = 0$

$E = 1, -2, 3$

$\therefore$ CF $= C_1 1^n + C_2(-2)^n + C_3 3^n$

2) Solve $U_{n+2} - 2U_{n+1} + U_n = 0$

Symbolic form of the given equation is

$(E^2 U_n - 2E U_n + U_n) = 0$

$(E^2 - 2E + 1)U_n = 0$

The auxilliary equation is

$E^2 - 2E + 1 = 0$

$E = 1, 1$

$\therefore$ CF $= (C_1 + C_2 n)1^n$

$= C_1 + C_2 n$

3) Solve $y_{n+1} - 2y_n\cos\alpha + y_{n-1} = 0$

Symbolic form of the given equation is

$(E^2 Y_{n-1} - 2E\cos\alpha\, Y_{n-1} + Y_{n-1}) = 0$

$(E^2 - 2E\cos\alpha + 1)Y_{n-1} = 0$

The auxilliary equation is

$E^2 - 2E\cos\alpha + 1 = 0$

$E = \cos\alpha + i\sin\alpha,\ \cos\alpha - i\sin\alpha$

C.F. $= y_n = r^n(C_1 \cos n\theta + C_2 \sin n\theta)$

$\theta = \tan^{-1}(\beta/\alpha) = \tan^{-1}(\sin\alpha / \cos\alpha) = \tan^{-1}(\tan\alpha) = \alpha$

$r = \sqrt{\alpha^2 + \beta^2} = \sqrt{\mathbf{cos^2}\theta + \sin^2\theta} = 1$

C.F $= y_{n-1} = C_1\cos(n-1)\alpha + C_2\sin(n-1)\alpha$

Rules for finding the particular integral

Consider the equation ,

$y_{n+r} + a_1 y_{n+r-1} + a_2 y_{n+r-2} + \ldots + a_r y_r = f(n)$

The symbolic form is,

$\Phi(E)\, y_n\, f(n) \ldots\ldots (1)$

Where $\Phi(E) = E^r + a_1 E^{r-1} + \ldots + a_r$

Then the particular integral is given by,

$P.I. = \frac{1}{\Phi(E)} f(n)$

Case 1

When $f(n) = a^n$

$P.I. = \frac{1}{\Phi(E)} a^n = \frac{1}{\Phi(a)} a^n$ provided $\Phi(a)$? 0 Replace E by a

If $\Phi(a) = 0$ then($E - a$) must be a factor of $\Phi(E)$ and we operate as follows,

$\frac{1}{E-a} a^n = n a^{n-1}$

$\frac{1}{(E-a)^2} a^n = n(n-1) a^{n-2}$ and so on.

Case 2

When $f(n) = n^p$

$P.I. = \frac{1}{\Phi(E)} n^p = \frac{1}{\Phi(1+\Delta)} n^p = [\Phi(1+\Delta)]^{-1} n^p$

Case 3

If $f(x) = \sin kn$ or $\cos kn$

$f(x) = \sin kn$

$P.I. = \frac{1}{\Phi(E)} \sin kn$

$= \frac{1}{\Phi(E)} \left[\frac{e^{ikn} - e^{-ikn}}{2!} \right]$

$= \frac{1}{2!} \left[\frac{1}{\Phi(E)} a^n - \frac{1}{\Phi(E)} b^n \right]$ Where $a = e^{ik}$ and $b = e^{-ik}$. Now proceed as in case1

$f(x) = \cos kn$

$P.I. = \frac{1}{\Phi(E)} \cos kn$

$= \frac{1}{\Phi(E)} \left[\frac{e^{ikn} + e^{-ikn}}{2} \right]$

$= \frac{1}{2} \left[\frac{1}{\Phi(E)} a^n - \frac{1}{\Phi(E)} b^n \right]$ Where $a = e^{ik}$ and $b = e^{-ik}$. Now proceed as in case 1

case 4

If $f(x) = a^n f(x)$ where $f(x)$ is a polynomial of degree p in x then

$P.I. = \frac{1}{\Phi(E)} a^n f(x) = a^n \frac{1}{\Phi(a.E)} f(x)$ Now proceed as in case 2

PROBLEMS

1) Solve $y_{n+2} - 4y_{n+1} + 3y_n = 5^n$

Symbolic form is $(E^2 - 4E + 3)\, y_n = 5^n$

The auxiliary equation is , $E^2 - 4E + 3 = 0$

$E = 1, 3$

C.F. = $C_1 1^n + C_2 3^n$

$$\text{P.I.} = \frac{1}{E^2 - 4E + 3} 5^n = \frac{1}{5^2 - 4*5 + 3} 5^n = \frac{5^n}{8}$$

The complete solution is $y_n = C_1 1^n + C_2 3^n + \frac{5^n}{8}$

2) Solve $y_{n+2} - 4y_{n+1} = n^2 + n - 1$

Symbolic form is $(E^2 - 4)\, y_n = n^2 + n - 1$

The auxiliary equation is , $E^2 - 4 = 0$

$E = 2, -2$

C.F. = $C_1 2^n + C_2 (-2)^n$

$$\text{P.I.} = \frac{1}{E^2 - 4} n^2 + n - 1$$

$$= \frac{1}{(1+\Delta)^2 - 4} \{[n]^2+2[n]-1\}$$

$$= \frac{1}{1+\Delta^2+2\Delta-4} \{[n]^2+2[n]-1\}$$

$$= \frac{1}{\Delta^2 + 2\Delta - 3} \{[n]^2+2[n]-1\}$$

$$= \frac{-1}{3} \cdot \frac{1}{\left(1 - \frac{\Delta^2+2\Delta}{3}\right)} \{[n]^2+2[n]-1\}$$

$$= -\frac{1}{3} \left[1 - \frac{2\Delta}{3} + \frac{\Delta^2}{3}\right]^{-1} \{[n]^2+2[n]-1\}$$

$$= -\frac{n^2}{3} - \frac{7n}{9} - \frac{17}{27}$$

The complete solution is ,

$$yn = c_1 2^n + c_2 (-2)^n - \frac{n^2}{3} - \frac{7n}{9} - \frac{17}{27}$$

Problems

1. Solve $U_{n+2} - 4U_{n+1} + 4U_n = 2^n$
2. Solve $y_{n+2} - 4y_n = 2^n$
3. Solve $u_{n+2} - 4U_{n+1} + U_n = 3$
4. $y_{x+2} + y_x = \sin x$
5. $y_{n+2} - 4y_{n+1} - 5y_n = 3^n + 5n + 8$

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