

**CMELR - Engineering Mathematics - 101 -  
2009-2010**

**Prepared by**

**Preetha Varghese  
Department of Science  
SNGCE Kadayiruppu  
Kolenchery - 682311**

## Fourier Series

### 5.1. PERIODIC FUNCTIONS

A function  $f(x)$  which satisfies the relation  $f(x+T) = f(x)$  for all  $x$  is called a periodic function. The smallest positive number  $T$ , for which this relation holds is called the **period** of  $f(x)$ .

If  $T$  is the period of  $f(x)$ , then  $f(x) = f(x+T) = f(x+2T) = \dots = f(x+nT) = \dots$

Also  $f(x) = f(x-T) = f(x-2T) = \dots = f(x-nT) = \dots$

$\therefore f(x) = f(x \pm nT)$ , where  $n$  is a positive integer.

Thus,  $f(x)$  repeats itself after periods of  $T$ .

For example,  $\sin x$ ,  $\cos x$ ,  $\sec x$  and  $\operatorname{cosec} x$  are periodic functions with period  $2\delta$  while  $\tan x$  and  $\cot x$  are periodic functions with period  $\delta$ . The functions  $\sin x$  and  $\cos nx$  are periodic with period  $\frac{2\delta}{n}$ .

### 5.2. FOURIER SERIES

Periodic functions are of common occurrence in many physical and engineering problems. For example, in conduction of heat and mechanical vibrations. It is useful to express these functions in a series of sines and cosines. Most of the single valued functions which occur in applied mathematics can be expressed in the form  $\frac{a_0}{2} + a_1 \cos x + a_2 \cos 2x + a_3 \cos 3x + \dots$

$b_1 \sin x + b_2 \sin 2x + b_3 \sin 3x + \dots$  within a desired range of values of the variable. Such a series is known as Fourier Series. Thus, any function  $f(x)$  defined in the interval  $c_1 \leq x \leq c_2$  can be expressed in the Fourier Series.

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

Where  $a_0, a_n, b_n$  ( $n = 1, 2, 3, \dots$ ) are constants, called the Fourier co-efficients of  $f(x)$ .

Note. To determine  $(a_0, a_n, b_n)$ , we shall need the following results: ( $m$  and  $n$  are integers)

$$(i) \int_c^{c+2\delta} \sin nx \, dx = - \left[ \frac{\cos nx}{n} \right]_c^{c+2\delta} = 0, \int_c^{c+2\delta} \cos nx \, dx = \left[ \frac{\sin nx}{n} \right]_c^{c+2\delta} = 0, n \neq 0$$

$$(ii) \int_c^{c+2\delta} \sin mx \cos nx \, dx = \frac{1}{2} \int_c^{c+2\delta} [\sin (m+n)x + \sin (m-n)x] \, dx \\ = - \frac{1}{2} \left[ \frac{\cos (m+n)x}{m+n} + \frac{\cos (m-n)x}{m-n} \right]_c^{c+2\delta} = 0, m \neq n$$

$$(iii) \int_c^{c+2\delta} \cos mx \cos nx \, dx = \frac{1}{2} \int_c^{c+2\delta} [\cos (m+n)x + \cos (m-n)x] \, dx \\ = \frac{1}{2} \left[ \frac{\sin (m+n)x}{m+n} + \frac{\sin (m-n)x}{m-n} \right]_c^{c+2\delta} = 0, m \neq n$$

$$(iv) \int_c^{c+2\delta} \sin mx \sin nx \, dx = \frac{1}{2} \int_c^{c+2\delta} [\cos (m-n)x - \cos (m+n)x] \, dx \\ = \frac{1}{2} \left[ \frac{\sin (m-n)x}{m-n} - \frac{\sin (m+n)x}{m+n} \right]_c^{c+2\delta} = 0, m \neq n$$

$$(v) \int_c^{c+2\delta} \cos^2 nx \, dx = \left[ \frac{x}{2} + \frac{\sin 2nx}{4n} \right]_c^{c+2\delta} = \delta, \int_c^{c+2\delta} \sin^2 nx \, dx = \left[ \frac{x}{2} - \frac{\sin 2nx}{4n} \right]_c^{c+2\delta} \\ = \delta, n \neq 0$$

$$(vi) \int_c^{c+2\delta} \sin nx \cos nx \, dx = \frac{1}{2} \int_c^{c+2\delta} \sin 2nx \, dx = - \frac{1}{2} \left[ \frac{\cos 2nx}{2n} \right]_c^{c+2\delta} = 0, n \neq 0$$

(vii) To integrate the product of two functions, one of which is a positive integral power of  $x$ , we apply the generalised rule of integration by parts. If dashes denote differentiation and suffixes denote integration w.r.t.  $x$ , the rule can be stated as follows:

$\int uv \, dx = uv^1 - u'v^2 + u''v^3 - u'''v^4 + \dots$  where  $u$  and  $v$  are functions of  $x$ , i.e.,  
Integral of the product of two functions.

= 1st function  $\times$  integral of 2nd - go on differentiating 1st, integrating 2nd, signs alternately +ve and -ve.

(Simplification should be done only when the integration is over)

(ix) Even and Odd Functions

A function  $f(x)$  is said to be even if  $f(-x) = f(x)$  e.g.  $x^2$ ,  $\cos x$ ,  $\sin^2 x$  are even functions.

The graph of an even function is symmetrical about the y- axis

A function  $f(x)$  is said to be odd if  $f(-x) = -f(x)$  eg.  $x^3$ ,  $\sin x$ ,  $\tan^3 x$  are odd functions

The graph of an odd function is symmetrical about the origin.

The product of two even functions or two odd functions is an even function while the product of an even function and an odd function is an odd function.

Also,  $\int_{-c}^c f(x) dx = 0$ , when  $f(x)$  is an odd function

and  $\int_c^c f(x) dx = 2 \int_0^c f(x) dx$ , when  $f(x)$  is an even function

### 5.3. EULER'S FORMULAE

The Fourier series for the function  $f(x)$  in the interval  $c < x < c + 2\delta$  is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx) \quad \dots(1)$$

In finding the co-efficients  $a_0$ ,  $a_n$  and  $b_n$  we assume that the series on the right hand side of (1) is uniformly convergent for  $c < x < c + 2\delta$  and it can be integrated term by term in the given interval.

**To find  $a_0$ ,** Integrate both sides of (1) w.r.t.x between the limits  $c$  to  $c + 2\delta$

$$\int_c^{c+2\delta} f(x) dx = \frac{a_0}{2} \int_c^{c+2\delta} dx + \int_c^{c+2\delta} \left[ \sum_{n=1}^{\infty} a_n \cos nx \right] dx + \int_c^{c+2\delta} \left[ \sum_{n=1}^{\infty} b_n \sin nx \right] dx$$

$$= \frac{a_0}{2} (c + 2\delta - c) + 0 + 0$$

$$= a_0 \delta$$

$$\therefore \boxed{a_0 = \frac{1}{\delta} \int_c^{c+2\delta} f(x) dx}$$

To find  $a_n$ , multiply both sides of (1) by  $\cos nx$  and integrate w.r.t.x between the limits  $c$  to  $c+2\delta$

$$\begin{aligned} \int_c^{c+2\delta} f(x) \cos nx \, dx &= \frac{a_0}{2} \int_c^{c+2\delta} \cos nx \, dx \\ &+ \int_c^{c+2\delta} \left[ \sum_{n=1}^{\infty} a_n \cos nx \right] \cos nx \, dx + \int_c^{c+2\delta} \left[ \sum_{n=1}^{\infty} b_n \sin nx \right] \cos nx \, dx \\ &= 0 + a_n \delta + 0 \qquad \qquad \qquad \text{[by formulae (i), (v) and vi]} \\ &= a_n \delta \end{aligned}$$

$$\therefore \boxed{a_n = \frac{1}{\delta} \int_c^{c+2\delta} f(x) \cos nx \, dx}$$

To find  $b_n$ , multiply both sides of (1) by  $\sin nx$  and integrate w.r.t.x between the limits  $c$  to  $c + 2\delta$

$$\begin{aligned} \int_c^{c+2\delta} f(x) \sin nx \, dx &= \frac{a_0}{2} \int_c^{c+2\delta} \sin nx \, dx + \int_c^{c+2\delta} \left[ \sum_{n=1}^{\infty} a_n \cos nx \right] \sin nx \, dx \\ &+ \int_c^{c+2\delta} \left[ \sum_{n=1}^{\infty} b_n \sin nx \right] \sin nx \, dx \\ &= 0 + 0 + b_n \delta \qquad \qquad \qquad \text{[by formulae (i), (vi) and v]} \\ &= b_n \delta \end{aligned}$$

$$\therefore \boxed{b_n = \frac{1}{\delta} \int_c^{c+2\delta} f(x) \sin nx \, dx}$$

Hence  $a_0 = \frac{1}{\delta} \int_c^{c+2\delta} f(x) \, dx$ ;  $a_n = \frac{1}{\delta} \int_c^{c+2\delta} f(x) \cos nx \, dx$ ; and  $b_n = \frac{1}{\delta} \int_c^{c+2\delta} f(x) \sin nx \, dx$

These values of  $a_0$ ,  $a_n$  and  $b_n$  are called Euler's formulae.

**Cor. 1.** If  $c = 0$ , the interval becomes  $0 < x < 2\delta$  and the formulae I reduce to

$$a_0 = \frac{1}{\delta} \int_0^{2\delta} f(x) \, dx; \quad a_n = \frac{1}{\delta} \int_0^{2\delta} f(x) \cos nx \, dx; \quad b_n = \frac{1}{\delta} \int_0^{2\delta} f(x) \sin nx \, dx$$

**Cor. 2.** If  $c = -\delta$ , the interval becomes  $-\delta < x < \delta$  and the formulae I reduce to

$$a_0 = \frac{1}{\delta} \int_{-\delta}^{\delta} f(x) dx; \quad a_n = \frac{1}{\delta} \int_{-\delta}^{\delta} f(x) \cos nx dx; \quad b_n = \frac{1}{\delta} \int_{-\delta}^{\delta} f(x) \sin nx dx$$

**Cor. 3.** When  $f(x)$  is an odd function.  $a_0 = \frac{1}{\delta} \int_{-\delta}^{\delta} f(x) dx = 0$

Since  $\cos nx$  is an even function, therefore,  $f(x) \cos nx$  is an odd function.

$$a_n = \frac{1}{\delta} \int_{-\delta}^{\delta} f(x) \cos nx dx = 0$$

Since  $\sin nx$  is an odd function, therefore,  $f(x) \sin nx$  is an even function.

$$\therefore b_n = \frac{1}{\delta} \int_{-\delta}^{\delta} f(x) \sin nx dx = \frac{2}{\delta} \int_0^{\delta} f(x) \sin nx dx$$

Hence, if a periodic function  $f(x)$  is odd, its Fourier expansion contains only sine terms,

$$\text{i.e., } f(x) = \sum_{n=1}^{\infty} b_n \sin nx, \text{ where } b_n = \frac{2}{\delta} \int_0^{\delta} f(x) \sin nx dx$$

When  $f(x)$  is an even function  $a_0 = \frac{1}{\delta} \int_{-\delta}^{\delta} f(x) dx = \frac{2}{\delta} \int_0^{\delta} f(x) dx$

Since  $\cos nx$  is an even function, therefore,  $f(x) \cos nx$  is an even function.

$$\therefore a_n = \frac{1}{\delta} \int_{-\delta}^{\delta} f(x) \cos nx dx = \frac{2}{\delta} \int_0^{\delta} f(x) \cos nx dx$$

Since  $\sin nx$  is an odd function, therefore,  $f(x) \sin nx$  is an odd function.

$$\therefore b_n = \frac{1}{\delta} \int_{-\delta}^{\delta} f(x) \sin nx dx = 0$$

Hence, if a periodic function  $f(x)$  is even, its Fourier expansion contains only cosine terms,

$$\text{i.e., } f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx, \text{ where } a_0 = \frac{2}{\delta} \int_0^{\delta} f(x) dx \text{ and } a_n = \frac{2}{\delta} \int_0^{\delta} f(x) \cos nx dx$$

H.W.

1. Expand  $f(x) = x \sin x$ ,  $0 < x < 2\delta$  as a Fourier series.

Sol. Let  $f(x) = x \sin x = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$

By Euler's formulae, we have  $a_0 = \frac{1}{\delta} \int_0^{2\delta} f(x) dx = \frac{1}{\delta} \int_0^{2\delta} x \sin x dx$

$$= \frac{1}{\delta} \left[ x(-\cos x) - 1 \cdot (-\sin x) \right]_0^{2\delta} = \frac{1}{\delta} [-2\delta] = -2$$

$$a_n = \frac{1}{\delta} \int_0^{2\delta} f(x) \cos nx dx = \frac{1}{\delta} \int_0^{2\delta} x \sin x \cos nx dx = \frac{1}{2\delta} \int_0^{2\delta} x (2 \cos nx \sin x) dx$$

$$= \frac{1}{2\delta} \int_0^{2\delta} x [\sin (n+1)x - \sin (n-1)x] dx$$

$$= \frac{1}{2\delta} \left[ x \left\{ -\frac{\cos (n+1)x}{n+1} + \frac{\cos (n-1)x}{n-1} \right\} - 1 \cdot \left\{ -\frac{\sin (n+1)x}{(n+1)^2} + \frac{\sin (n-1)x}{(n-1)^2} \right\} \right]_0^{2\delta}$$

$$= \frac{1}{2\delta} \left[ 2\delta \left\{ -\frac{\cos 2(n+1)\delta}{n+1} + \frac{\cos 2(n-1)\delta}{n-1} \right\} \right]$$

$$= -\frac{1}{n+1} + \frac{1}{n-1} = \frac{2}{n^2-1}, \quad n \neq 1$$

When  $n = 1$ , we have  $a_1 = \frac{1}{\delta} \int_0^{2\delta} x \sin x \cos x dx = \frac{1}{2\delta} \int_0^{2\delta} x \sin 2x dx$

$$= \frac{1}{2\delta} \left[ x - \frac{(-\cos 2x)}{2} - 1 \cdot \frac{(-\sin 2x)}{4} \right]_0^{2\delta} = \frac{1}{2\delta} [-\delta] = -\frac{1}{2}$$

$$b_n = \frac{1}{\delta} \int_0^{2\delta} f(x) \sin nx dx = \frac{1}{\delta} \int_0^{2\delta} x \sin x \sin nx dx$$

$$= \frac{1}{2\delta} \int_0^{2\delta} x (2 \sin nx \sin x) dx = \frac{1}{2\delta} \int_0^{2\delta} x [\cos(n-1)x - \cos(n+1)x] dx$$

$$= \frac{1}{2\delta} \left[ x \left\{ \frac{\sin (n-1)x}{n-1} - \frac{\sin (n+1)x}{n+1} \right\} - 1 \cdot \left\{ -\frac{\cos (n-1)x}{(n-1)^2} + \frac{\cos (n+1)x}{(n+1)^2} \right\} \right]_0^{2\delta}$$

$$\begin{aligned}
&= \frac{1}{2\delta} \left[ \frac{\cos 2(n-1)\delta}{(n-1)^2} - \frac{\cos 2(n+1)\delta}{(n+1)^2} - \frac{1}{(n-1)^2} + \frac{1}{(n+1)^2} \right] \\
&= \frac{1}{2\delta} \left[ \frac{1}{(n-1)^2} - \frac{1}{(n+1)^2} - \frac{1}{(n+1)^2} + \frac{1}{(n+1)^2} \right] = 0, n \neq 1
\end{aligned}$$

When  $n = 1$ , we have  $b_1 = \frac{1}{\delta} \int_0^{2\delta} x \sin x \sin x \, dx = \frac{1}{2\delta} \int_0^{2\delta} x (1 - \cos 2x) \, dx$

$$\begin{aligned}
&= \frac{1}{2\delta} x \left[ x - \frac{\sin 2x}{2} \right] - \frac{1}{2} \left[ \frac{(x)^2}{2} + \frac{\cos 2x}{2} \right]_0^{2\delta} \\
&= \frac{1}{2\delta} \left[ 2\delta (2\delta) - \frac{4\delta^2}{2} - \frac{1}{4} + \frac{1}{4} \right] = \frac{1}{2\delta} (2\delta)^2 = \delta
\end{aligned}$$

$$\begin{aligned}
\therefore f(x) &= \frac{a_0}{2} + a_1 \cos x + b_1 \sin x + \sum_{n=2}^{\infty} a_n \cos nx + \sum_{n=2}^{\infty} b_n \sin nx \\
&= -1 - \frac{1}{2} \cos x + \delta \sin x + \sum_{n=2}^{\infty} \frac{2}{n^2-1} \cos nx + 0 \\
&= -1 + \delta \sin x - \frac{1}{2} \cos x + \frac{2}{2^2-1} \cos 2x + \frac{2}{3^2-1} \cos 3x + \dots
\end{aligned}$$

Example. 3. Find a Fourier series to represent  $x-x^2$  from  $x = -\delta$  to  $x = \delta$ . Hence show that

$$\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \dots = \frac{\delta^2}{12}$$

Sol. Let  $x - x^2 = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$

By Euler's formulae, we have  $a_0 = \frac{1}{\delta} \int_{-\delta}^{\delta} (x-x^2) \, dx = \frac{1}{\delta} \left[ \frac{x^2}{2} - \frac{x^3}{3} \right]_{-\delta}^{\delta}$

$$= \frac{1}{\delta} \left[ \frac{\delta^2}{2} - \frac{\delta^3}{3} \right] - \left[ \frac{\delta^2}{2} + \frac{\delta^3}{3} \right] = -\frac{2\delta^2}{3}$$

$$a_n = \frac{1}{\delta} \int_{-\delta}^{\delta} (x-x^2) \cos nx \, dx$$

$$= \frac{1}{\delta} \left[ (x-x^2) \frac{\sin nx}{n} - (1-2x) \left( -\frac{\cos nx}{n^2} \right) + (-2) \left( -\frac{\sin nx}{n^2} \right) \right]_{-\delta}^{\delta}$$



$$= \frac{1}{\delta} \left[ (1-2\delta) \frac{\cos n\delta}{n^2} - (1+2\delta) \frac{\cos n\delta}{n^2} \right] = \frac{1}{\delta} \left( -4\delta \cdot \frac{\cos n\delta}{n^2} \right)$$

$$= -4 \frac{(-1)^2}{n^2}$$

$$b_n = \frac{1}{\delta} \int_{-\delta}^{\delta} (x-x)^2 \sin nx \, dx$$

$$= \frac{1}{\delta} \left[ (x-x)^2 \left( -\frac{\cos nx}{n} \right) - (1-2x) \left( -\frac{\sin nx}{n^2} \right) + (-2) \left( \frac{\cos nx}{n^3} \right) \right]_{-\delta}^{\delta}$$

$$= \frac{1}{\delta} \left[ (\delta^2-\delta) \frac{\cos n\delta}{n} - 2 \frac{\cos n\delta}{n^3} + (-\delta-\delta) \frac{\cos n\delta}{n} + 2 \frac{\cos n\delta}{n^3} \right]$$

$$= \frac{1}{\delta} \left[ -2\delta \cdot \frac{\cos n\delta}{n} \right] = -2 \frac{(-1)^n}{\delta}$$

$$\therefore x-x^2 = -\frac{\delta^2}{3} - 4 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos n\delta - 2 \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \sin nx$$

$$= -\frac{\delta^2}{3} - 4 \left[ -\frac{\cos x}{1^2} + \frac{\cos 2x}{2^2} - \frac{\cos 3x}{3^2} + \dots \right]$$

$$- 2 \left[ -\frac{\sin x}{1} + \frac{\sin 2x}{2} - \frac{\sin 3x}{3} + \dots \right]$$

$$= -\frac{\delta^2}{3} + 4 \left[ \frac{\cos x}{1^2} - \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} - \dots \right] + 2 \left[ \frac{\sin x}{1} - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \dots \right]$$

Putting  $x = 0$ , we get  $0 = -\frac{\delta^2}{3} + 4 \left[ \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \dots \right]$

$$\Rightarrow \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \dots = \frac{\delta^2}{12}$$

**Example 4.** Obtain the Fourier series for  $f(x) = e^{-x}$  in the interval  $0 < x < 2\delta$

Sol. Let  $f(x) = e^{-x} = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$

$$\text{Then } a_0 = \frac{1}{\delta} \int_0^{2\delta} f(x) dx = \frac{1}{\delta} \int_0^{2\delta} e^{-x} dx = \left[ \frac{1}{\delta} e^{-x} \right]_0^{2\delta} = \frac{1 - e^{-2\delta}}{\delta}$$

$$a_n = \frac{1}{\delta} \int_0^{2\delta} f(x) \cos nx dx = \frac{1}{\delta} \int_0^{2\delta} e^{-x} \cos nx dx$$

$$= \frac{1}{\delta} \left[ \frac{e^{-x}}{1+n^2} (-\cos nx + n \sin nx) \right]_0^{2\delta}$$

$$= \frac{1-e^{-2\delta}}{\delta(1+n^2)} \left[ \because \int e^{ax} \cos bx dx = \frac{e^{ax}}{a^2+b^2} (a \cos nx + n \sin nx) \right]$$

$$b_n = \frac{1}{\delta} \int_0^{2\delta} f(x) \sin nx dx = \frac{1}{\delta} \int_0^{2\delta} e^{-x} n \sin nx dx$$

$$= \frac{1}{\delta} \left[ \frac{e^{-x}}{1+n^2} (-\sin nx - n \cos nx) \right]_0^{2\delta} = \frac{1-e^{-2\delta}}{\delta} \cdot \frac{n}{1+n^2}$$

$$\left[ \because \int e^{ax} \sin bx dx = \frac{e^{ax}}{a^2+b^2} (a \sin bx - b \cos bx) \right]$$

$$\therefore e^{-x} = \frac{1-e^{-2\delta}}{2\delta} + \frac{1-e^{-2\delta}}{\delta} \sum_{n=1}^{\infty} \frac{\cos nx}{1+n^2} + \frac{1-e^{-2\delta}}{\delta} \sum_{n=1}^{\infty} \frac{n \sin nx}{1+n^2}$$

$$= \frac{1-e^{-2\delta}}{\delta} \left[ \frac{1}{2} + \left( \frac{1}{2} \cos x + \frac{1}{5} \cos 2x + \frac{1}{10} \cos 3x + \dots \right) \right. \\ \left. + \left( \frac{1}{2} \sin x + \frac{2}{5} \sin 2x + \frac{3}{10} \sin 3x + \dots \right) \right]$$

**Example 5.** Express  $f(x) = |x|$ ,  $-\delta < x < \delta$ , as Fourier series.

Sol. Since  $f(x) = |-x| = |x| = f(x)$

$\therefore f(x)$  is an even function and hence  $b_n = 0$

Let 
$$f(x) = x = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

Then 
$$a_0 = \frac{2}{\delta} \int_0^{\delta} f(x) dx = \frac{2}{\delta} \int_0^{\delta} |x| dx = \frac{2}{\delta} \int_0^{\delta} x dx = \frac{2}{\delta} \left[ \frac{x^2}{2} \right]_0^{\delta} = \delta$$

$$\begin{aligned} a_n &= \frac{2}{\delta} \int_0^{\delta} f(x) \cos nx dx = \frac{2}{\delta} \int_0^{\delta} |x| \cos nx dx = \frac{2}{\delta} \int_0^{\delta} x \cos nx dx \\ &= \frac{2}{\delta} \left[ x \frac{\sin nx}{n} - 1 \cdot \left( \frac{-\cos nx}{n^2} \right) \right]_0^{\delta} \\ &= \frac{2}{\delta} \left[ \frac{\cos n\delta}{n^2} - \frac{1}{n^2} \right] = \frac{2}{\delta n^2} [(-1)^n - 1] \begin{cases} 0, & \text{if } n \text{ is even} \\ -4, & \text{if } n \text{ is odd} \end{cases} \end{aligned}$$

$$\therefore |x| = \frac{\delta}{2} - \frac{4}{\delta} \left( \cos x + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \dots \right)$$

Note. Putting  $x = 0$  in the above result, we get  $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots = \frac{\delta^2}{8}$

**Example 10.** Obtain Fourier series for the function  $f(x)$  given by

$$f(x) \begin{cases} 1 + \frac{2x}{\delta}, & -x \leq x \leq 0 \\ 1 - \frac{2x}{\delta}, & 0 \leq x \leq \delta \end{cases}$$

Hence deduce that  $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots = \frac{\delta^2}{8}$

Sol. When  $-\delta \leq x \leq 0$ ,  $0 \leq -x \leq \delta$

$$\therefore f(-x) = 1 - \frac{2(-x)}{\delta} = 1 + \frac{2x}{\delta} = f(x)$$

When  $0 \leq x \leq \delta$ ,  $-\delta \leq x \leq 0$

$$\therefore f(-x) = 1 - \frac{2(-x)}{\delta} = 1 - \frac{2x}{\delta} = f(x)$$

$$1 - \frac{2x}{\delta}$$

$\Rightarrow f(x)$  is an even function of  $x$  in  $[-\delta, \delta]$ . This is also clear from its graph which is symmetrical above the  $y$ -axis.

$$\therefore b_n = 0$$

Let 
$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

Then 
$$a_0 = \frac{2}{\delta} \int_0^{\delta} f(x) dx = \frac{2}{\delta} \int_0^{\delta} \left(1 - \frac{2x}{\delta}\right) dx = \frac{2}{\delta} \left[ x - \frac{x^2}{\delta} \right]_0^{\delta} = 0$$

$x dx =$

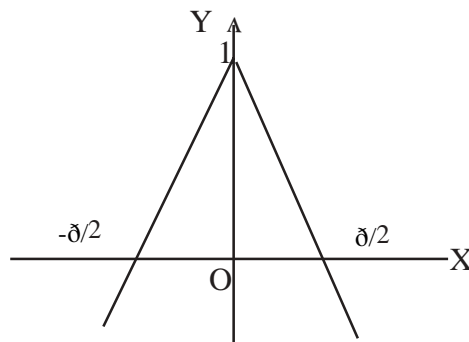
$$\begin{aligned} a_n &= \frac{2}{\delta} \int_0^{\delta} f(x) \cos nx dx = \frac{2}{\delta} \int_0^{\delta} \left(1 - \frac{2x}{\delta}\right) \cos nx dx \\ &= \frac{2}{\delta} \left[ 1 - \frac{2x}{\delta} \cdot \frac{\sin nx}{n} - \left(\frac{2}{\delta}\right) \left(-\frac{\cos nx}{n^2}\right) \right]_0^{\delta} \\ &= \frac{2}{\delta} \left[ -\frac{2 \cos n\delta}{\delta n^2} + \frac{2}{\delta n^2} \right] = \frac{4}{\delta^2 n^2} [1 - (-1)^n] \end{aligned}$$

$$\begin{aligned} \therefore f(x) &= \frac{4}{\delta^2} \sum_{n=1}^{\infty} [1 - (-1)^n] \frac{\cos nx}{n^2} \\ &= \frac{4}{\delta^2} \left( \frac{2 \cos x}{1^2} + \frac{2 \cos 3x}{3^2} + \frac{2 \cos 5x}{5^2} + \dots \right) \end{aligned}$$

[  $\therefore 1 - (-1)^n = 0$  when  $n$  is even ]

$$= \frac{8}{\delta^2} \left( \frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \dots \right)$$

Putting  $x = 0$ , we get 
$$\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots = \frac{\delta^2}{8}, \text{ since } f(0) = 1$$



## H.W

1. Expand in a Fourier series the function  $f(x) = x$  in the interval  $0 < x < 2\delta$ .
2. Express  $f(x) = 1/2 (\delta - x)$  in a Fourier series in the interval  $0 < x < 2\delta$ .
3. Find the Fourier series for the function  $f(x) = x + x^2$  in the interval  $-\delta < x < \delta$ .

Hence show that

$$(i) \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \dots = \frac{\delta^2}{12}$$

$$(ii) 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \dots = \frac{\delta^2}{6}$$

4. If  $f(x) = \left(\frac{\delta - x}{2}\right)^2$  in the interval  $0 < x < 2\delta$ , show that  $f(x) = \frac{\delta^2}{12} + \sum_{n=1}^{\infty} \frac{\cos nx}{n^2}$

Hence obtain the following relations:

$$(i) \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \dots = \frac{\delta^2}{6}$$

$$(ii) \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \dots = \frac{\delta^2}{12}$$

$$(iii) \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots = \frac{\delta^2}{8}$$

5. Prove that for all values of  $x$  between  $-\delta$  and  $\delta$   $\frac{1}{2}x = \sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - \frac{1}{4}\sin 4x + \dots$
6. Obtain the Fourier series to represent  $e^x$  in the interval  $0 < x < 2\delta$
7. Find the Fourier series to represent  $e^x$  in the interval  $-\delta < x < \delta$
8. Find the Fourier series to represent the function  $f(x) = |\sin x|$ ,  $-\delta < x < \delta$
9. Expand  $f(x) = \cos x$  as a Fourier series in the interval  $-\delta < x < \delta$

10. Prove that in the interval  $-\delta < x < \delta$ ,  $x \cos x = -\frac{1}{2} \sin x + 2 \sum_{n=2}^{\infty} \frac{(-1)^n}{n^2-1} \sin nx$

11. Prove that for  $-\delta < x < \delta$ ,  $\frac{x(\delta^2 - x^2)}{12} = \frac{\sin x}{1^3} - \frac{\sin 2x}{2^3} + \frac{\sin 3x}{3^3} - \frac{\sin 4x}{4^3} + \dots$

12.(a) Obtain a Fourier expansion for  $\sqrt{1-\cos x}$  in the interval  $-\delta < x < \delta$ .  
Hint. For all integral values of  $n$ ,  $\cos(n + \frac{1}{2})\delta = 0 = \cos(n - \frac{1}{2})\delta$

(b) Obtain a Fourier series for  $\sqrt{1-\cos x}$  in the interval  $(0, 2\delta)$  and hence find the value of

$$\frac{1}{1.3} + \frac{1}{3.5} + \frac{1}{5.7} + \dots$$

13. Express  $f(x) = \cos wx$ ,  $-\delta < x < \delta$ , where  $w$  is a fraction, as a Fourier series. Hence prove that  $\cot \theta = \frac{1}{\theta} + \frac{2\theta}{\theta^2 - \delta^2} + \frac{2\theta}{\theta^2 - 4\delta^2} + \dots$

14. Find the Fourier series for  $f(x)$  in the interval  $(-\delta, \delta)$  when

$$\begin{aligned} f(x) &= \delta + \delta, & -\delta < x < 0 \\ &= \delta - x, & 0 < x < \delta. \end{aligned}$$

15. Obtain a Fourier series to represent  $e^{-ax}$  from  $x = -\delta$  to  $x = \delta$ . Hence derive series for

$$\frac{\delta}{\sinh \delta}$$

16. Prove that in the range  $-\delta < x < 0$ ,  $\cosh ax = \frac{2a}{\delta} \sin h a\delta \left[ \frac{1}{2a^2} + \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2+a^2} \cos nx \right]$

17. Given  $f(x) = \begin{cases} -x+1 & \text{for } -\delta \leq x \leq 0 \\ x+1 & \text{for } 0 \leq x \leq \delta \end{cases}$

Is the function even or odd? Find the Fourier series for  $f(x)$  and deduce the value of

$$\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots$$

18. Find the Fourier series for  $f(x) = x+x^2$ ;  $-\delta < x < \delta$ .

## ANSWERS

1.  $f(x) = \delta - 2 \sum_{n=1}^{\infty} \frac{\sin nx}{n}$

2.  $f(x) = \sum_{n=1}^{\infty} \frac{\sin nx}{n}$

3.  $f(x) = \frac{\delta^3}{3} - 4 \left( \frac{\cos x}{12} - \frac{\cos 2x}{22} + \frac{\cos 3x}{32} - \dots \right) + 2 \left( \frac{\sin x}{1} - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \dots \right)$

6.  $e^x = \frac{e^{2\delta}-1}{2\delta} + \frac{e^{2\delta}-1}{\delta} \sum_{n=1}^{\infty} \left( \frac{\cos nx}{1+n^2} - \frac{n}{1+n^2} \sin nx \right)$

7.  $e^x = \frac{2 \sinh \delta}{\delta}$

8.  $|\sin x| = \frac{2}{\delta} - \frac{4}{\delta} \left( \frac{\cos 2x}{3} + \frac{\cos 4x}{15} + \dots + \frac{\cos 2nx}{4n^2-1} + \dots \right)$

9.  $|\cos x| = \frac{2}{\delta} + \frac{4}{\delta} \left( \frac{\cos 2x}{3} - \frac{\cos 4x}{15} + \dots \right)$

12. (a)  $\sqrt{1-\cos x} = \frac{2\sqrt{2}}{\delta} - \frac{4\sqrt{2}}{\delta} \sum_{n=1}^{\infty} \frac{\cos nx}{4n^2-1}$  (b). Same as in part (a);  $\frac{1}{2}$

13.  $\cos \omega x = \frac{2\omega \sin \omega \delta}{\delta} \left( \frac{1}{2\omega^2} + \frac{\cos x}{12-\omega^2} - \frac{\cos 2x}{22-\omega^2} + \frac{\cos 3x}{32-\omega^2} - \dots \right)$

14.  $f(x) = \frac{\delta}{2} + \frac{4}{\delta} \left( \frac{\cos x}{12} + \frac{\cos 3x}{32} + \frac{\cos 5x}{52} + \dots \right)$

15.  $e^{-ax} = \frac{2 \sinh a\delta}{\delta} \left[ \left( \frac{1}{2a} - \frac{a \cos x}{12+a^2} + \frac{a \cos 2x}{22+a^2} - \dots \right) - \left( \frac{\sin x}{12+a^2} - \frac{2 \sin 2x}{22+a^2} + \frac{3 \sin 3x}{32+a^2} - \dots \right) \right]$

$$\frac{\delta}{\sinh \delta} = 2 \left[ \frac{1}{22+1} - \frac{1}{32+1} + \frac{1}{42+1} - \dots \right]$$

17. Even  $f(x) = \frac{\delta}{2} + 1 - \frac{4}{\delta} \left( \cos x + \frac{\cos 3x}{32} + \frac{\cos 5x}{52} + \dots \right); \frac{\delta 2}{8}$

18.  $\frac{\delta 2}{8} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2} \cos nx + \sum_{n=1}^{\infty} \frac{2}{n} (-1)^{n+1} \sin nx.$

cos x  
12

δ

## 5.4. DIRICHLET'S CONDITIONS

The sufficient conditions for the uniform convergence of a Fourier series are called Dirichlet's conditions (after Dirichlet, a German mathematician). All the functions that normally arises in engineering problems satisfy these conditions and hence they can be expressed as a Fourier series.

Any function  $f(x)$  can be expressed as a Fourier series  $\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$

Where  $a_0, a_n, b_n$  are constants, provided

- (i)  $f(x)$  is periodic, single valued and finite
- (ii)  $f(x)$  has a finite number of finite discontinuities in any one period.
- (iii)  $f(x)$  has a finite number of maxima and minima

When these conditions are satisfied, the Fourier series converges to  $f(x)$  at every point of continuity. At a point of discontinuity, the sum of the series is equal to the mean of the limits on the right and left.

i.e,  $\frac{1}{2} [f(x+0) + f(x-0)]$

Where  $f(x+0)$  and  $f(x-0)$  denote the limit on the right and the limit on the left respectively.

## 5.5 FOURIER SERIES FOR DISCONTINUOUS FUNCTIONS

In 5.3, we derived Euler's formulae for  $a_0, a_n, b_n$  on the assumption that  $f(x)$  is continuous in  $(c, c+2\delta)$ . However, if  $f(x)$  has finitely many points of finite discontinuity, even then it can be expressed as a Fourier series. The integrals for  $a_0, a_n, b_n$  are to be evaluated by breaking up the range of integration.

Let  $f(x)$  be defined by  $f(x) = f_1(x), c < x < x_0$   
 $= f_2(x), x_0 < x < c+2\delta$

Where  $x_0$  is the point of finite discontinuity in the interval  $(c, c+2\delta)$ .

The values of  $a_0, a_n, b_n$  are given by

$$a_0 = \frac{1}{\delta} \left[ \int_c^{x_0} f_1(x) dx + \int_{x_0}^{c+2\delta} f_2(x) dx \right]$$

$$a_n = \frac{1}{\delta} \left[ \int_c^{x_0} f_1(x) \cos nx dx + \int_{x_0}^{c+2\delta} f_2(x) \cos nx dx \right]$$



$$b_n = \frac{1}{\delta} \left[ \int_c^{x_0} f_1(x) \sin nx \, dx + \int_{x_0}^{c+2\delta} f_2(x) \sin nx \, dx \right]$$

At  $x = x_0$ , there is a finite jump in the graph of the function. Both the limits  $f(x_0-0)$  and  $f(x_0+0)$  exist but are unequal. The sum of the Fourier series  $= \frac{1}{2} [f(x_0-0) + f(x_0+0)] = \frac{1}{2} (AB + AC) = AM$ , where M is the mid-point of BC.

### ILLUSTRATIVE EXAMPLES

Example 1. Find the Fourier series to represent the function  $f(x)$  given by

$$f(x) = x \quad \text{for } 0 \leq x \leq \delta$$

$$= 2\delta - x \quad \text{for } \delta \leq x \leq 2\delta$$

Deduce that  $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots = \frac{\delta^2}{8}$

Sol. Let  $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx +$

Then  $a_0 = \frac{1}{\delta} \int_0^{2\delta} f(x) \, dx = \frac{1}{\delta} \left[ \int_0^{\delta} x \, dx + \int_0^{2\delta} (2\delta - x) \, dx \right]$

$$= \frac{1}{\delta} \left[ \left. \frac{x^2}{2} \right|_0^{\delta} + \left. \left( 2\delta x - \frac{x^2}{2} \right) \right|_{\delta}^{2\delta} \right] = \frac{1}{\delta} \left[ \frac{\delta^2}{2} + (4\delta^2 - 2\delta^2) - \left( 2\delta^2 - \frac{\delta^2}{2} \right) \right] = \delta$$

$$a_n = \frac{1}{\delta} \int_0^{2\delta} f(x) \cos nx \, dx = \frac{1}{\delta} \left[ \int_0^{\delta} x \cos nx \, dx + \int_0^{2\delta} (2\delta - x) \cos nx \, dx \right]$$

$$= \frac{1}{\delta} \left[ \left. x \frac{\sin nx}{n} - 1 \cdot \left( -\frac{\cos nx}{n^2} \right) \right|_0^{\delta} + \left. \left( 2\delta - x \right) \left( \frac{\sin nx}{n} \right) - (-1) \left( -\frac{\cos nx}{n^2} \right) \right|_{\delta}^{2\delta} \right]$$

$$= \frac{1}{\delta} \left[ \left( \frac{\cos n\delta}{n^2} - \frac{1}{n^2} \right) + \left( -\frac{\cos 2n\delta}{n^2} + \frac{\cos n\delta}{n^2} \right) \right]$$

$$= \frac{1}{\delta n^2} [(-1)^{n-1} - 1 + (-1)^n] = \frac{2}{\delta n^2} [(-1)^{n-1}] = \begin{cases} -\frac{4}{\delta n^2}, & \text{if } n \text{ is odd} \\ 0, & \text{if } n \text{ is even} \end{cases}$$

$$\begin{aligned}
b_n &= \frac{1}{\delta} \int_0^{2\delta} f(x) \sin nx \, dx = \frac{1}{\delta} \left[ \int_0^{\delta} x \sin nx \, dx + \int_{\delta}^{2\delta} (2\delta-x) \sin nx \, dx \right] \\
&= \frac{1}{\delta} \left[ \left[ x \left( \frac{-\cos nx}{n} \right) - 1 \cdot \left( \frac{-\sin nx}{n^2} \right) \right]_0^{\delta} \right. \\
&\quad \left. + \left[ (2\delta-x) x \left( \frac{-\cos nx}{n} \right) - 1 \cdot \left( \frac{-\sin nx}{n^2} \right) \right]_{\delta}^{2\delta} \right] \\
&= \frac{1}{\delta} \left[ -\frac{\delta \cos n\delta}{n} + \frac{\delta \cos n\delta}{n} \right] = 0 \\
\therefore f(x) &= \frac{\delta}{2} - \frac{4}{\delta} \left( \frac{\cos x}{12} + \frac{\cos 3x}{32} + \frac{\cos 5x}{52} + \dots \right) \\
\text{Putting } x=0, \text{ we get } 0 &= \frac{\delta}{2} - \frac{4}{\delta} \left( \frac{1}{12} + \frac{1}{32} + \frac{1}{52} + \dots \right) \\
&\Rightarrow \frac{\delta^2}{8} = \frac{1}{12} + \frac{1}{32} + \frac{1}{52} + \dots
\end{aligned}$$

Example 2. If  $f(x) = 0$ ,  $-\delta \leq x \leq 0$   
 $= \sin x$ ,  $0 \leq x \leq \delta$ ,

Prove that  $f(x) = \frac{1}{\delta} + \frac{1}{2} \sin x - \frac{2}{\delta} \sum_{n=1}^{\infty} \frac{\cos 2nx}{4n^2-1}$

Hence show that

(i)  $\frac{1}{1.3} + \frac{1}{3.5} + \frac{1}{5.7} + \dots = \frac{1}{2}$

(ii)  $\frac{1}{1.3} - \frac{1}{3.5} + \frac{1}{5.7} - \dots = \frac{\delta-2}{4}$

Sol. Let  $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$

Then  $a_0 = \frac{1}{\delta} \int_{-\delta}^{\delta} f(x) \, dx = \frac{1}{\delta} \left[ \int_{-\delta}^0 0 \, dx + \int_0^{\delta} \sin x \, dx \right] = \frac{2}{\delta}$

$$a_n = \frac{1}{\delta} \int_{-\delta}^{\delta} f(x) \cos nx \, dx = \frac{1}{\delta} \left[ \int_{-\delta}^0 0 \, dx + \int_0^{\delta} \sin x \cos nx \, dx \right]$$

$$= \frac{1}{2\delta} \int_0^{\delta} 2 \cos nx \sin x \, dx = \frac{1}{2\delta} \int_0^{\delta} [\sin (n+1)x - \sin (n-1)x] \, dx$$

0 dx +

$$\begin{aligned}
&= \frac{1}{2\delta} \left[ -\frac{\cos(n+1)x}{n+1} + \frac{\cos(n-1)x}{n-1} \right]_0^\delta, n \neq 1 \\
&= \frac{1}{2\delta} \left[ -\frac{\cos(n+1)\delta}{n+1} + \frac{\cos(n-1)\delta}{n-1} + \frac{1}{n+1} - \frac{1}{n-1} \right] \\
&= \frac{1}{2\delta} \left[ -\frac{(-1)^{n+1}}{n+1} + \frac{(-1)^{n-1}}{n-1} + \frac{1}{n+1} - \frac{1}{n-1} \right]
\end{aligned}$$

$$\begin{aligned}
&= \begin{cases} \frac{1}{2\delta} \left( -\frac{1}{n+1} + \frac{1}{n-1} + \frac{1}{n+1} - \frac{1}{n-1} \right), & \text{when } n \text{ is odd} \\ \frac{1}{2\delta} \left( \frac{1}{n+1} - \frac{1}{n-1} + \frac{1}{n+1} - \frac{1}{n-1} \right), & \text{when } n \text{ is even} \end{cases} \\
&= \begin{cases} 0, & \text{when } n \text{ is odd, i.e., } n = 3, 5, 7, \dots \\ -\frac{2}{\delta(n^2+1)}, & \text{when } n \text{ is even} \end{cases}
\end{aligned}$$

When  $n = 1$ , we have  $a_1 = \frac{1}{\delta} \int_0^\delta \sin x \cos x \, dx = \frac{1}{2\delta} \int_0^\delta \sin 2x \, dx = \frac{1}{2\delta} \left[ -\frac{\cos 2x}{2} \right]_0^\delta = 0$

$$\begin{aligned}
b_n &= \frac{1}{\delta} \int_{-\delta}^\delta f(x) \sin nx \, dx = \frac{1}{\delta} \left[ \int_{-\delta}^0 0 \, dx + \int_0^\delta \sin x \sin nx \, dx \right] \\
&= \frac{1}{2\delta} \int_0^\delta 2 \sin nx \sin x \, dx = \frac{1}{2\delta} \int_0^\delta [\cos(n-1)x - \cos(n+1)x] \, dx \\
&= \frac{1}{2\delta} \left[ \frac{\sin(n-1)x}{n-1} - \frac{\sin(n+1)x}{n+1} \right]_0^\delta = 0, n \neq 1
\end{aligned}$$

When  $n = 1$ , we have

$$b_1 = \frac{1}{\delta} \int_0^\delta \sin x \sin x \, dx = \frac{1}{2\delta} \int_0^\delta (1 - \cos 2x) \, dx = \frac{1}{2\delta} \left[ x - \frac{\cos 2x}{2} \right]_0^\delta = \frac{1}{2}$$

$$\therefore f(x) = \frac{1}{\delta} - \frac{2}{\delta} \left[ \frac{\cos 2x}{2^2-1} + \frac{\cos 4x}{4^2-1} + \frac{\cos 6x}{6^2-1} + \dots \right] = \frac{1}{2} \sin x$$

$$= \frac{1}{\delta} + \frac{1}{2} \sin x - \frac{2}{\delta} \sum_{n=1}^{\infty} \frac{\cos 2nx}{(2n)^2-1} \quad \dots(1) \quad 0$$

Putting  $x = 0$  in (1) we have  $0 = \frac{1}{\delta} - \frac{2}{\delta} \sum_{n=1}^{\infty} \frac{1}{4n^2-1}$

$$\Rightarrow \frac{1}{2} = \sum_{n=1}^{\infty} \frac{1}{4n^2-1} = \sum_{n=1}^{\infty} \frac{1}{(2n-1)(2n+1)} = \frac{1}{1.3} + \frac{1}{3.5} + \frac{1}{5.7} + \dots$$

Putting  $x = \frac{\delta}{2}$  in (1) we have  $1 = \frac{1}{\delta} + \frac{1}{2} - \frac{2}{\delta} \sum_{n=1}^{\infty} \frac{\cos n\delta}{4n^2-1}$

$$\Rightarrow \frac{1}{2} - \frac{1}{\delta} = -\frac{2}{\delta} \sum_{n=1}^{\infty} \frac{(-1)^n}{4n^2-1}$$

$$\Rightarrow \frac{\delta-2}{4} = -\sum_{n=1}^{\infty} \frac{(-1)^n}{(2n-1)(2n+1)} = -\left(-\frac{1}{1.3} + \frac{1}{3.5} + \frac{1}{5.7} + \dots\right)$$

$$\Rightarrow \frac{1}{1.3} - \frac{1}{3.5} + \frac{1}{5.7} - \dots = \frac{\delta-2}{4}$$

## EXERCISE 5.2

1. Find the Fourier series to represent the function

$$f(x) = -k \quad \text{when } -\delta < x < 0$$

$$= k \quad \text{when } 0 < x < \delta$$

Also deduce that  $\frac{\delta}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots$

2. Develop  $f(x)$  in a Fourier series in the interval  $(-\delta, \delta)$  if  $f(x) = 0$  when  $-\delta < x < 0$   
 $= 1$  when  $0 < x < \delta$
3. Find the Fourier series expansion for  $f(x)$ , if  $f(x) = -\delta - \delta < x < 0$   
 $= x, 0 < x < \delta$

Deduce that  $\frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots = \frac{\delta-2}{8}$

$f(x) =$

Hint. For the deduction, put  $x=0$  in the expansion of  $f(x)$ .

$$f(0-0) = -\delta \text{ and } f(0+0) = 0 \quad \therefore f(0) = \frac{1}{2} [f(0-0) + f(0+0)] = -\frac{\delta}{2}$$

4. Find the Fourier expansion of the function defined in one period by the relations

$$f(x) = 1 \text{ for } 0 < x < \delta$$

$$= 2 \text{ for } \delta < x < 2\delta$$

and deduce that  $\frac{\delta}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots$

$$-\delta \leq x \leq 0$$

$$0 \leq x \leq \delta,$$

5. Find the Fourier series of the following function:

$$f(x) = x^2 \quad 0 \leq x \leq \delta$$

$$= -x^2 \quad -\delta \leq x \leq 0$$

6. An alternating current after passing through a rectifier has the form

$$i = I_0 \sin x \quad \text{for} \quad 0 \leq x \leq \delta,$$

$$= 0 \quad \text{for} \quad \delta \leq x \leq 2\delta$$

$$= -x^2$$

Where  $I_0$  is the maximum current and the period is  $2\delta$ . Express  $i$  as a Fourier series.

7. Obtain Fourier series for the function

$$f(x) = \begin{cases} x & \text{for } -\delta \leq x \leq 0 \\ -x & \text{for } 0 \leq x \leq \delta, \end{cases}$$

and hence show that

$$\frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots = \frac{\delta-2}{8}$$

8. Find the Fourier series for the function

$$f(x) = \begin{cases} -1 & \text{for } -\delta < x < -\frac{\delta}{2} \\ 0 & \text{for } -\frac{\delta}{2} < x < \frac{\delta}{2} \\ 1 & \text{for } \frac{\delta}{2} < x < \delta \end{cases}$$

### Answers

1.  $f(x) = \frac{4k}{\delta} \left( \sin x + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \dots \right)$

2.  $f(x) = \frac{1}{2} + \frac{2}{\delta} \left( \sin x + \frac{\sin 5x}{5} + \dots \right)$

3.  $f(x) = -\frac{\delta}{4} - \frac{2}{\delta} \left( \cos x + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \dots \right) + \left( 3 \sin x - \frac{\sin 2x}{2} + \sin 3x - \frac{\sin 4x}{4} + \dots \right)$

4.  $f(x) = \frac{3}{2} - \frac{2}{\delta} \left( \sin x + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \dots \right)$

5.  $f(x) = 2\left(\delta - \frac{4}{\delta}\right) \sin x - \delta \sin 2x + \frac{2}{3}\left(\delta - \frac{4}{9\delta}\right) \sin 3x - \frac{\delta}{2} \sin 4x + \dots$

6.  $i = \frac{I_0}{2} + \frac{I_0}{2} \sin x - \frac{2I_0}{\delta} \left( \frac{\cos 2x}{2^2-1} + \frac{\cos 4x}{4^2-1} + \frac{\cos 6x}{6^2-1} + \dots \right)$

7.  $f(x) = -\frac{\delta}{2} + \frac{4}{\delta} \left( \frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \dots \right)$

8.  $f(x) = \frac{2}{\delta} \left( \sin x - \sin 2x + \frac{\sin 3x}{3} - \dots \right)$

## 5.6 CHANGE OF INTERVAL

In many engineering problems, it is desired to expand a function in a Fourier series over an interval of length  $2l$  and not  $2\delta$ . In order to apply foregoing theory, this interval must be transformed into an interval of length  $2\delta$ . This can be achieved by a transformation of the variable.

Consider a periodic function  $f(x)$  defined in the interval  $c < x < c+2l$ . To change the interval into one of length  $2\delta$ , we put

$$\frac{x}{l} = \frac{z}{\delta} \quad \text{or} \quad = \frac{\delta x}{l} \quad \text{so that when } x = c, z = \frac{\delta c}{l} = d \text{ (say)}$$

$$\text{and when } x = c+2l, \quad z = \frac{\delta(c+2l)}{l} = \frac{\delta c}{l} + 2\delta = d + 2\delta$$

Thus the function  $f(x)$  of period  $2l$  in  $(c, c+2l)$  is transformed to the function  $f\left(\frac{lz}{\delta}\right) = F(z)$ , say, of period  $2\delta$  in  $(d, d+2\delta)$  and the latter function can be expressed as the Fourier series.

$$F(z) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nz + \sum_{n=1}^{\infty} b_n \sin nz \quad \dots(1)$$

$$\text{Where } a_0 = \frac{1}{\delta} \int_d^{d+2\delta} F(z) dz; \quad a_n = \frac{1}{\delta} \int_d^{d+2\delta} F(z) \cos nz dz; \quad \text{and } b_n = \frac{1}{\delta} \int_d^{d+2\delta} F(z) \sin nz dz \dots (2)$$

$$\text{Now making the inverse substitution } z = \frac{\delta x}{l}, \quad dz = \frac{\delta x}{l} dx$$

When  $z = d$ ,  $x = c$  and when  $z = d+2\delta$ ,  $x = c+2l$ .

$$\text{The expression (1) becomes } F(z) = F\left(\frac{\delta x}{l}\right) f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\delta x}{l} + \sum_{n=1}^{\infty} b_n \sin \frac{n\delta x}{l}$$

$\frac{n\delta x}{l}$  and the coefficients  $a_0, a_n, b_n$  from (2) reduce to

$$a_0 = \frac{1}{l} \int_c^{c+2l} f(x) dx; \quad a_n = \frac{1}{l} \int_c^{c+2l} f(x) \cos \frac{n\delta x}{l} dx; \quad \text{and } b_n = \frac{1}{l} \int_c^{c+2l} f(x) \sin \frac{n\delta x}{l} dx$$

Hence the Fourier series for  $f(x)$  in the interval  $c < x < c+2l$  is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\delta x}{l} + \sum_{n=1}^{\infty} b_n \sin \frac{n\delta x}{l}$$

Where  $a_0 = \frac{1}{l} \int_c^{c+2l} f(x) dx$ ,  $a_n = \frac{1}{l} \int_c^{c+2l} f(x) \cos \frac{n\delta x}{l} dx$  and  $b_n = \frac{1}{l} \int_c^{c+2l} f(x) \sin \frac{n\delta x}{l} dx$ .

**Cor. 1.** If we put  $c = 0$ , the interval becomes  $0 < x < 2l$  and the above results reduce to

$$a_0 = \frac{1}{l} \int_0^{2l} f(x) dx; \quad a_n = \frac{1}{l} \int_0^{2l} f(x) \cos \frac{n\delta x}{l} dx; \quad \text{and} \quad b_n = \frac{1}{l} \int_0^{2l} f(x) \sin \frac{n\delta x}{l} dx.$$

**Cor. 2.** If we put  $c = -l$ , the interval becomes  $-l < x < l$  and the above results reduce to

$$a_0 = \frac{1}{l} \int_{-l}^l f(x) dx; \quad a_n = \frac{1}{l} \int_{-l}^l f(x) \cos \frac{n\delta x}{l} dx; \quad \text{and} \quad b_n = \frac{1}{l} \int_{-l}^l f(x) \sin \frac{n\delta x}{l} dx.$$

**If  $f(x)$  is an even function,** we have

$$a_0 = \frac{2}{\delta} \int_0^l f(x) dx, \quad a_n = \frac{2}{l} \int_0^l f(x) \cos \frac{n\delta x}{l} dx \quad \text{and} \quad b_n = 0$$

**If  $f(x)$  is an odd function,** we have  $a_0 = 0, a_n = 0$

$$\text{and } b_n = \frac{2}{l} \int_0^l f(x) \sin \frac{n\delta x}{l} dx.$$

**Example 1.** Find the Fourier series to represent  $f(x) = x^2 - 2$ , when  $-2 \leq x \leq 2$ .

Sol. Since  $f(x)$  is an even function,  $b_n = 0$

$$\text{Let } f(x) = x^2 - 2 = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\delta x}{2}$$

$$\text{Then } a_0 = \frac{2}{2} \int_0^2 (x^2 - 2) dx = \left[ \frac{x^3}{3} - 2x \right]_0^2 = \frac{8}{3} - 4 = -\frac{4}{3}$$

$$a_n = \frac{2}{2} \int_0^2 (x^2 - 2) \cos \frac{n\delta x}{2} dx$$

$$\left[ (x^2 - 2) \sin \frac{n\delta x}{2} - 2x \left[ \frac{\cos n\delta x}{\frac{n^2 \delta^2}{4}} \right] + 2 \left[ \frac{\cos \frac{n\delta x}{2}}{\frac{n^3 \delta^3}{8}} \right] \right]_0^2$$

$$= \frac{16 \cos n\delta}{n^2 \delta^2} = \frac{16 (-1)^n}{n^2 \delta^2}$$

$$\therefore x^2 - 2 = -\frac{2}{3} - \frac{16}{\delta^2} \left[ \cos \frac{\delta x}{2} - \frac{1}{4} \cos \delta x + \frac{1}{9} \cos \frac{3\delta x}{2} - \dots \right]$$

### EXERCISE 5.3

1. Find a Fourier series for  $f(t) = 1 - t^2$  when  $-1 \leq t \leq 1$
2. Expand  $f(x)$  in Fourier series in the interval  $(-2, 2)$  when  $f(x) = 0, -2 < x < 0$   
 $= 1, 0 < x < 2$
3. Develop  $f(x)$  in a Fourier series in the interval  $(0, 2)$  if  $f(x) = x, 0 < x < 1$   
 $= 0, 1 < x < 2$
4. Find the Fourier expansion for  $f(x) = \delta x$  from  $x = -c$  to  $x = c$ .
5. Find the Fourier expansion for the function  $f(x) = x - x^3$  in the interval  $-1 < x < 1$ .
6. Find the Fourier series for the function given by  $f(t) = t, 0 < t < 1$   
 $= 1 - t, 1 < t < 2$ .
7. Find a Fourier series to represent  $x^2$  in the interval  $(-l, l)$
8. Expand  $f(x) = e^{-x}$  as a Fourier series in the interval  $(-2, 2)$

9. Expand:  $f(x) = \begin{cases} \frac{1}{4} - x, & 0 < x < \frac{1}{2} \\ x - \frac{3}{4}, & \frac{1}{2} < x < 1 \\ 1, & 1 < x < \frac{3}{2} \\ x - 1, & \frac{3}{2} < x < 2 \end{cases}$

as a Fourier series.

10. Find the Fourier series for the function  $f(x) = \begin{cases} 0 & \text{when } -2 < x < -1 \\ k & \text{when } -1 < x < 1 \\ 0 & \text{when } 1 < x < 2 \end{cases}$
11. A sinusoidal voltage  $E \sin \omega t$  is passed through a half-wave rectifier which clips the negative portion of the wave.

Expand the resulting periodic function  $u(t) = \begin{cases} 0 & \text{when } -\frac{T}{2} < t < 0 \\ E \sin \omega t & \text{when } 0 < t < \frac{T}{2} \end{cases}$

and  $T = \frac{2\pi}{\omega}$  in a Fourier series



## 5.7. HALF RANGE SERIES

Sometimes it is required to expand a function  $f(x)$  in the range  $(0, \delta)$  in a Fourier series of period  $2\delta$  or more generally in the range  $(0, l)$  in a Fourier series of period  $2l$ .

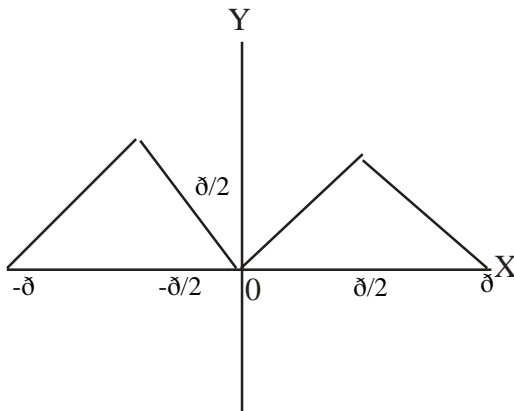
If it is required to expand  $f(x)$  in the interval  $(0, l)$ , then it is immaterial what the function may be outside range  $0 < x < l$ . We are free to choose it arbitrarily in the interval  $(-l, 0)$

If we extend the function  $f(x)$  by reflecting it in the y-axis so that  $f(-x) = f(x)$ , then the extended function is even for which  $b_n = 0$ . The Fourier expansion of  $f(x)$  will contain only cosine terms.

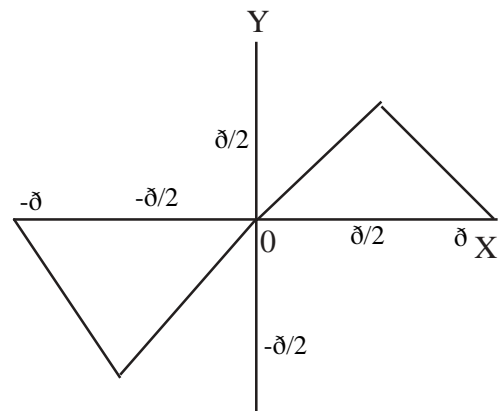
If we extend the function  $f(x)$  by reflecting it in the origin so that  $f(-x) = -f(x)$ , then the extended function is odd for which  $a_0 = a_n = 0$ . The Fourier expansion of  $f(x)$  will contain only sine terms.

For example, consider the function

$$f(x) = \begin{cases} x & 0 < x < \frac{\delta}{2} \\ \delta - x & \frac{\delta}{2} < x < \delta \end{cases}$$



(Reflection in the y-axis)



(Reflection in the origin)

Hence a function  $f(x)$  defined over the interval  $0 < x < l$  is capable of two distinct half-range series.

The half range cosine series is  $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\delta x}{l}$

where  $a_0 = \frac{2}{l} \int_0^1 f(x) dx$ ;  $a_n = \frac{2}{l} \int_0^1 f(x) \cos \frac{n\delta x}{l} dx$

The half - range sine series is  $f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\delta x}{l}$

where  $b_n = \frac{2}{l} \int_0^1 f(x) \sin \frac{n\delta x}{l} dx$

**Cor.** If the range is  $0 < x < \delta$ , then

(i) The half - range cosine series is  $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$

where  $a_0 = \frac{2}{\delta} \int_0^{\delta} f(x) dx$ ;  $a_n = \frac{2}{\delta} \int_0^{\delta} f(x) \cos nx dx$

(ii) The half - range sine series is  $f(x) = \sum_{n=1}^{\infty} b_n \sin nx$ , where  $b_n = \frac{2}{\delta} \int_0^{\delta} f(x) \sin nx dx$

**Example 1.** Expand  $\delta x - x^2$  in a half - range sine series in the interval  $(0, \delta)$  up to the first three terms.

Sol. Let  $\delta x - x^2 = \sum_{n=1}^{\infty} b_n \sin nx$ , then

$$\begin{aligned} b_n &= \frac{2}{\delta} \int_0^{\delta} (\delta x - x^2) \sin nx dx \\ &= \frac{2}{\delta} \left[ (\delta x - x^2) \left( -\frac{\cos nx}{n} \right) - (\delta - 2x) \left( -\frac{\cos nx}{n^2} \right) + (-2) \left( \frac{\cos nx}{n^3} \right) \right]_0^{\delta} \\ &= \frac{2}{\delta} \left[ -\frac{\cos n\delta}{n^3} + \frac{2}{n^3} \right] = \frac{4}{\delta n^2} [1 - (-1)^n] \\ &= 0 \text{ or } \frac{8}{\delta n^3} \text{ according as } n \text{ is even or odd} \end{aligned}$$

$$\therefore \delta x - x^2 = \frac{8}{\delta} \left( \sin x + \frac{\sin 3x}{3^3} + \frac{\sin 5x}{5^3} + \dots \right)$$

**Example 2.** Express  $f(x) = x(\delta - x)$ ;  $0 < x < \delta$  as Fourier sine series in period  $2\delta$ .

Sol. The Fourier sine series expansion of  $f(x)$  is  $f(x) = \sum_{n=1}^{\infty} b_n \sin nx$

where

$$b_n = \frac{2}{\delta} \int_0^{\delta} f(x) \sin nx \, dx$$

Now

$$\begin{aligned} b_n &= \frac{2}{\delta} \int_0^{\delta} (\delta x - x^2) \sin nx \, dx \\ &= \frac{2}{\delta} \left[ (\delta x - x^2) \frac{\cos nx}{n} - (\delta - 2x) \frac{\sin nx}{n^2} + (-2) \frac{\cos nx}{n^3} \right]_0^{\delta} \\ &= \frac{2}{\delta} \frac{(-2) \cos n\delta}{n^3} - \frac{2}{\delta} \frac{(-2)}{n^3} \\ &= \frac{-4}{\delta n^3} (\cos n\delta - 1) \\ &= \frac{-4}{n^3 \delta} [(-1)^n - 1] \end{aligned}$$

Therefore the series expansion is  $f(x) = \sum_{n=1}^{\infty} \frac{-4}{n^3 \delta} [(-1)^n - 1] \sin nx$

Hence the result.

### EXERCISE 5.4

1.(a) Obtain cosine and sine series for  $f(x) = x$  in the interval  $0 \leq x \leq \delta$ . Hence show that

$$\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots = \frac{\delta^2}{8}$$

(b) Prove that for  $0 < x < l$

$$x = \frac{l}{2} - \frac{4l}{\delta^2} \left( \cos \frac{\delta x}{l} + \frac{1}{3^2} \cos \frac{3\delta x}{l} + \frac{1}{5^2} \cos \frac{5\delta x}{l} + \dots \right)$$

2. Find the half-range cosine series for the function  $f(x) = x^2$  in the range  $0 \leq x \leq \delta$ .

3. Find the half-range cosine series for the function  $f(x) = (x-1)^2$  in the interval  $0 < x < 1$ .

Hence show that

$$(i) \quad \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \dots = \frac{\delta^2}{6}$$

$$(ii) \quad \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \dots = \frac{\delta^2}{12}$$

$$(iii) \quad \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \dots = \frac{\delta^2}{8}$$

- 4.(a) Express  $\sin x$  as a cosine series in  $0 < x < \delta$ .  
 (b) Show that a constant function  $c$  can be expanded in an infinite series

$$\frac{4}{\delta} \left( \sin x + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \dots \right) \text{ in the range } 0 < x < \delta.$$

5. If 
$$f(x) = \frac{\delta}{3} \quad 0 \leq x \leq \frac{\delta}{3}$$

$$= 0 \quad \frac{\delta}{3} \leq x \leq \frac{2\delta}{3}$$

$$= -\frac{\delta}{3}, \quad \frac{2\delta}{3} \leq x \leq \delta$$

then show that 
$$f(x) = \frac{2}{\sqrt{3}} \left[ \cos x - \frac{\cos 5x}{5} + \frac{\cos 7x}{7} - \dots \right]$$

6. If 
$$f(x) = mx, \quad 0 \leq x \leq \frac{\delta}{2}$$

$$= m(\delta - x), \quad \frac{\delta}{2} \leq x \leq \delta$$

then show that 
$$f(x) = \frac{4m}{\delta} \left[ \frac{\sin x}{1^2} - \frac{\sin 3x}{3^2} + \frac{\sin 5x}{5^2} - \dots \right]$$

7. Express  $f(x) = x$  as a half range

- (i) sine series in  $0 < x < 2$ .      (ii) Cosine series in  $0 < x < 2$ .

8. Show that the series  $\frac{1}{\delta} \sum_{n=1}^{\infty} \frac{1}{n} \sin \frac{2n\delta x}{l}$  represents  $\frac{1}{2} l - x$  when  $0 < x < l$

9. Find the half-range sine series for  $f(x) = \frac{1}{4} - x, \quad 0 < x < \frac{1}{2}$   

$$= x - \frac{3}{4}, \quad \frac{1}{2} < x < 1.$$

10. Represent the following function by Fourier sine series

$$f(x) = 1 \text{ when } 0 < x < \frac{1}{2}$$

$$= 0 \text{ when } \frac{1}{2} < x <$$

11. Find the half-range sine series for the function  $f(t) = t - t^2, 0 < t < 1$

12. Prove that for  $0 < x < \delta$ ,

$$= \frac{\delta^2}{6} - \left( \frac{\cos 2x}{1^2} + \frac{\cos 4x}{2^2} + \frac{\cos 6x}{3^2} + \dots \right)$$

13. Let  $f(x) = \begin{cases} \omega x, & \text{when } 0 \leq x \leq \frac{l}{2} \\ \omega(l-x), & \text{when } \frac{l}{2} \leq x \leq l \end{cases}$

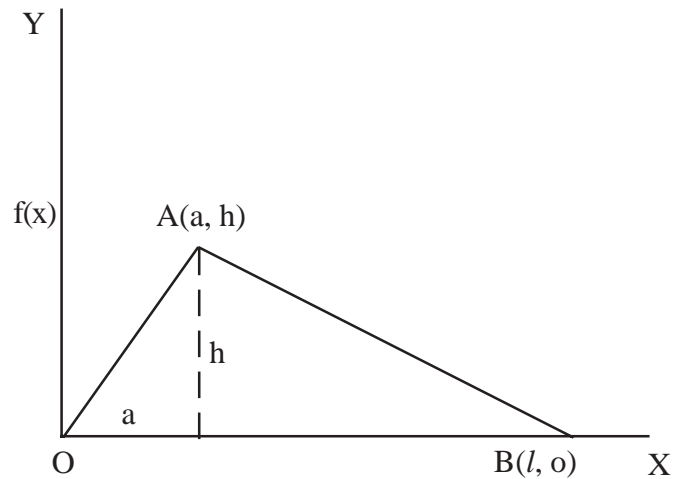
Show that  $f(x) = \frac{4\omega l}{\delta^2} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2} \sin \frac{(2n+1)\delta x}{l}$

Hence obtain the sum of the series

$$1 + \frac{1}{3^2} + \frac{1}{5^2} + \dots$$

14. If  $f(x) = \begin{cases} \sin x, & \text{for } 0 \leq x < \frac{\delta}{4} \\ \cos x, & \text{for } \frac{\delta}{4} \leq x \leq \frac{\delta}{2} \end{cases}$  expand  $f(x)$  in a series of sines

15. For the function defined by the graph OAB, find the half-range Fourier sine series.



16. Find the half-range sine series for  $f(x) = \frac{1}{2} - x$ ,  $0 < x < l$

17. Find the half-range cosine series for  $f(x) = x \sin x$ ,  $0 < x < \delta$

## 5.8. PARSEVAL'S THEOREM ON FOURIER CONSTANTS

If the Fourier series of  $f(x)$  over an interval  $c < x < c+2l$  is given as

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left\{ a_n \cos \frac{n\check{x}}{l} + b_n \sin \frac{n\check{x}}{l} \right\}$$

$$\text{then } \frac{1}{2l} \int_c^{c+2l} [f(x)]^2 dx = \frac{a_0^2}{4} + \frac{1}{2} \sum_{n=1}^{\infty} (a_n^2 + b_n^2)$$

Proof. The Fourier series of  $f(x)$  in  $c < x < c+2l$  is given as

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left\{ a_n \cos \frac{n\check{x}}{l} + b_n \sin \frac{n\check{x}}{l} \right\}$$

$$\text{where } a_0 = \frac{1}{l} \int_c^{c+2l} f(x) dx; \quad a_n = \frac{1}{l} \int_c^{c+2l} f(x) \cos \frac{n\check{x}}{l} dx; \quad b_n = \frac{1}{l} \int_c^{c+2l} f(x) \sin \frac{n\check{x}}{l} dx$$

Multiplying both sides of (1) by  $f(x)$ , we have

$$[f(x)]^2 = \frac{a_0}{2} f(x) + \sum_{n=1}^{\infty} a_n f(x) \cos \frac{n\check{x}}{l} + \sum_{n=1}^{\infty} b_n f(x) \sin \frac{n\check{x}}{l}$$

Integrating both sides w.r.t.  $x$  between the limits  $c$  to  $c + 2l$ , we have

$$\begin{aligned} \int_c^{c+2l} [f(x)]^2 dx &= \frac{a_0}{2} \int_c^{c+2l} f(x) dx + \sum_{n=1}^{\infty} a_n \int_c^{c+2l} f(x) \cos \frac{n\check{x}}{l} dx \\ &\quad + \sum_{n=1}^{\infty} b_n \int_c^{c+2l} f(x) \sin \frac{n\check{x}}{l} dx \\ &= \frac{a_0}{2} \cdot la_0 + \sum_{n=1}^{\infty} a_n (la_n) + \sum_{n=1}^{\infty} b_n (lb_n) \end{aligned}$$

$$\int_c^{c+2l} [f(x)]^2 dx = \frac{la_0^2}{2} + l \sum_{n=1}^{\infty} (a_n^2 + b_n^2)$$

$$\therefore \frac{1}{2l} \int_c^{c+2l} [f(x)]^2 dx = \frac{1}{2l} \left\{ \frac{la_0^2}{2} + l \sum_{n=1}^{\infty} (a_n^2 + b_n^2) \right\}$$

$$\text{or } \frac{1}{2l} \int_c^{c+2l} [f(x)]^2 dx = \frac{a_0^2}{4} + \frac{1}{2} \sum_{n=1}^{\infty} (a_n^2 + b_n^2) \quad (\text{Parseval's identity})$$

Hence the proof

Note: Parseval's identities in different cases:

(i) If  $c = 0$ , the interval becomes  $0 < x < 2l$  and Parseval's identity reduces to

$$\frac{1}{2l} \int_0^{2l} [f(x)]^2 dx = \frac{a_0^2}{4} + \frac{1}{2} \sum_{n=1}^{\infty} (a_n^2 + b_n^2)$$

If  $c = -l$ , the interval becomes  $-l < x < l$  and Parseval's identity reduces to

$$\frac{1}{2l} \int_{-l}^l [f(x)]^2 dx = \frac{a_0^2}{4} + \frac{1}{2} \sum_{n=1}^{\infty} (a_n^2 + b_n^2)$$

(ii) If  $f(x)$  is an even function in  $(-l, l)$  then  $\frac{2}{l} \int_0^l [f(x)]^2 dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty} a_n^2$

(iii) If  $f(x)$  is an odd function in  $(-l, l)$  then  $\frac{2}{l} \int_0^l [f(x)]^2 dx = \sum_{n=1}^{\infty} b_n^2$

(iv) If  $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{l}$  in  $(0, l)$  then  $\frac{2}{l} \int_0^l [f(x)]^2 dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty} a_n^2$

(v) If  $f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}$  in  $(0, l)$  then  $\frac{2}{l} \int_0^l [f(x)]^2 dx = \sum_{n=1}^{\infty} b_n^2$

### 5.9. ROOT MEAN SQUARE (r.m.s.) VALUE

The root mean square value of the function  $f(x)$  over an interval  $(a, b)$  is defined as

$$\left[ f(x) \right]_{(r.m.s)} = \sqrt{\frac{\int_a^b [f(x)]^2 dx}{b-a}}$$

The r.m.s value of a function is also known as the effective value of the function.

## ILLUSTRATIVE EXAMPLES

**Example 1.** Find the Fourier sine series for unity in  $0 < x < \delta$  and hence show that

$$1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \dots = \frac{\delta^2}{8}$$

Sol. We require half-range Fourier sine series for 1 in  $(0, \delta)$

Let 
$$1 = \sum_{n=1}^{\infty} b_n \sin nx$$

Then 
$$b_n = \frac{2}{\delta} \int_0^{\delta} (1) \sin nx \, dx = \frac{2}{\delta} \left[ -\frac{\cos nx}{n} \right]_0^{\delta} = -\frac{2}{n\delta} (\cos n\delta - 1)$$

$$= \frac{2}{n\delta} [1 - (-1)^n]$$

$$\therefore [\cos n\delta = (-1)^n]$$

Now  $b_n = 0$  when  $n$  is even, and  $b_n = \frac{4}{n\delta}$  when  $n$  is odd.

Substituting in (1), we get

$$\therefore 1 = \sum_{m=1}^{\infty} \frac{4}{(2m-1)\delta} \sin (2m-1)x \text{ or } 1 = \frac{4}{\delta} \left( \sin x + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \dots \right)$$

Now from Parseval's theorem on Fourier constants

$$\int_c^{c+2l} [f(x)]^2 dx = 2l \left[ \frac{a_0^2}{4} + \frac{l}{2} \sum_{n=1}^{\infty} (a_n^2 + b_n^2) \right]$$

Applying (3) to half-range sine series for 1 in  $(0, \delta)$

$c = 0, 2l = \delta, f(x) = 1, a_0 = 0, a_n = 0$  and  $b_n = \frac{4}{(2m-1)\delta}, m = 1, 2, \dots$

We get, 
$$\int_0^{\delta} (1)^2 dx = \delta \cdot \frac{1}{2} \sum_{n=1}^{\infty} \frac{16}{(2m-1)^2} \cdot \delta^2$$

$$\Rightarrow \left[ x \right]_0^{\delta} = \frac{8}{\delta} \left\{ \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots \right\} \text{ or } \frac{\delta^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \dots$$

Hence the result



Example 2. Find Fourier series of  $x^2$  in  $(-\delta, \delta)$ . Use Parseval's identity to prove that

$$\frac{\delta^4}{90} = 1 + \frac{1}{2^4} + \frac{1}{3^4} + \dots$$

Sol. The Fourier series of  $x^2$  in  $(-\delta, \delta)$  is

$$x^2 = \frac{\delta^2}{3} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2} \cos nx$$

Here  $a_0 = \frac{2\delta^2}{3}$ ,  $a_n = \frac{4(-1)^n}{n^2}$ ,  $b_n = 0$ ,  $f(x) = x^2$

Now by Parseval's identity from (1), we get

$$\int_{-\delta}^{\delta} (x^2)^2 dx = 2\delta \left[ \frac{\delta^4}{9} + \frac{1}{2} \sum_{n=1}^{\infty} \frac{16}{n^4} \right]$$

$$\Rightarrow \left[ \frac{x^5}{5} \right]_{-\delta}^{\delta} = \frac{2\delta^5}{9} + \delta \sum_{n=1}^{\infty} \frac{16}{n^4} \text{ or } \frac{2\delta^5}{5} - \frac{2\delta^5}{9} = \delta \sum_{n=1}^{\infty} \frac{16}{n^4}$$

or  $\frac{2\delta^5}{9} = \sum_{n=1}^{\infty} \frac{1}{n^4}$  or  $1 + \frac{1}{2^4} + \frac{1}{3^4} + \dots = \frac{\delta^4}{90}$ .

### EXERCISE 5.5

1. If  $f(x)$  has the Fourier series expansion

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} \left( a_n \cos \frac{n\delta x}{l} + b_n \sin \frac{n\delta x}{l} \right) \text{ in } a \leq x \leq a + 2l$$

show that  $\int_a^{a+2l} [f(x)]^2 dx = 2l \left[ \frac{a_0^2}{4} + \frac{l}{2} \sum_{n=1}^{\infty} (a_n^2 + b_n^2) \right]$

2. If  $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\delta x}{l}$  in  $0 < x < l$ , then show that

$$\int_0^l [f(x)]^2 dx = \frac{l}{2} \left[ \frac{a_0^2}{2} + \sum_{n=1}^{\infty} a_n^2 \right]$$

3. If  $f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\delta x}{l}$  in  $(0, l)$  then show that  $\int_0^l [f(x)]^2 dx = \frac{l}{2} \sum_{n=1}^{\infty} b_n^2$ .

4. Prove that in the range  $(0, l)$ ,  $x = \frac{l}{2} - \frac{4l}{\delta^2} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} \cos \frac{(2n-1)\delta x}{l}$

and deduce that

$$\frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \dots = \frac{\delta^4}{96}.$$

5. Show that for  $0 < x < \delta$ ,

$$x(\delta-x) = \frac{\delta^2}{6} - \left( \frac{\cos 2x}{1^2} + \frac{\cos 4x}{2^2} + \frac{\cos 6x}{3^2} + \dots \right)$$

and hence evaluate  $\sum_{n=1}^{\infty} \frac{1}{n^4}$ .