Dynamics.

Vibrations or oscillation → Subset of Dynamics

→ Subjected to the presence of a restoring force
due to elasticity or due to gravity

Response to a stimulus (restoring force).
Predicting Response is difficult but essential part of engineering systems.

→ Simplified model → Experimental

Mathematical choice of model depends on → use

→ System Mass
→ Stiffness properties
→ Parameters

→ Simulation of model by varying parameters

Behaviour is described in single (degree) differential eqn of motion

Set of Ordinary D.E → multiple degree of freedom → called discrete

Systems or lumped parameter systems.

Systems with → distributed mass → Stiffness → distributed parameter models

All behaviours described by partial diff eqn

Response of a system to a given excitation → depends on system characteristics.

Linear: response is proportional to the excitation

Nonlinear: 

Linearity → dictate the approach to the solution of eqns of motion

→ Principle of Superposition can be applied

In linear systems: 

Response due to initial excitation → can be calculated

Response due to external excitation → can be calculated separately and combined by p. of

Response to initial excitation → decay with time = called transient superposition

Sinoidal excitations: responses are treated in frequency domain than in

Time domain:

Periodic excitations: → represented as Sin functions by Fourier Series

Response: → Represented as corresponding Sin function → steady state

In both cases, time has no role.
Arbitrary excitations $\rightarrow$ given as impulses of varying magnitude
Laplace Transformation method used
Response to arbitrary excitations $\rightarrow$ obtained numerically $\rightarrow$ discrete time
Random excitations $\rightarrow$ have different approach
Response to Random excitation $\rightarrow$ obtained by statistical quantities

Multi degree of freedom systems + distributed parameter systems
Require more complex

Eigen values, them by direct Newton's laws.
Linear systems' eqns are expressed in Matrix form $\rightarrow$ kry of
Simultaneous eqns; the coefficient matrices are fully populated due to
its symmetry, $\rightarrow$ Solution of an algebraic eigenvalue problem,
$\Rightarrow$ called Modal analysis $\rightarrow$ Orthogonal transform formation using
Modal Matrix $\rightarrow$ linear algebra.
partial differential eqns are solved by a differential eigenvalue
problem instead of algebraic one.
$\Rightarrow$ Not analytical
$\Rightarrow$ hence approximate,
by ordering to linear
Structural dynamics $\left\{ \begin{array}{l}
\text{Rayleigh Ritz method} \\
\text{Galerkin method} \\
\text{discretization} \\
\text{methods}
\end{array} \right.$

For nonlinear systems $\Rightarrow$ perturbation techniques.

Longer computation.

In Random excitation, $\Rightarrow$ The response Runge-Kutta method.
also Random $\Rightarrow$ defined by statistical quantities,
$\Rightarrow$ Gaussian Random process
$\Rightarrow$ Probabilistic approach $\Rightarrow$ Mean value
standard deviation.
Oscillation → To end the motion with Restoring force.
Vibration → For mechanical systems, the oscillatory motion $a \rightarrow$ vibration

Newtonian Mechanics

Component Modelling

System Modelling

Derivation of System: D.E. of motion —— Newtonian mechanics

Analytical dynamics (Lagrangian Mechanics)

General excitation & Response characteristics

Motion stability

**Newton's Law**

1) 1st Law: $F = 0$, $\vec{v} = \text{constant}$.

2) 2nd Law

$$\vec{F} = \frac{dp}{dt} = \frac{d(m\vec{v})}{dt} = m\ddot{\vec{v}} = m\ddot{x} = ma$$

3) 3rd Law, $\vec{f}_{ij} = -\vec{f}_{ji}$, $i \neq j$

Moment of a force: $\vec{M}_o = \vec{r} \times \vec{F}$

Angular Momentum: $\vec{H}_0 = \vec{r} \times \vec{p} = \vec{r} \times m\vec{\omega}$

$$\vec{H}_0 = \vec{r} \times m\vec{\omega} + m\vec{r} \times \vec{\omega} = m(\vec{r} \times \vec{\omega}) + \vec{r} \times m\vec{\omega}$$

$$= \vec{r} \times \vec{p} = \vec{M}_0$$

If $\vec{M}_0 = 0$, $\vec{H}_0 = \text{constant}$, $m\vec{r} = \text{constant}$, $\vec{H}_0 = \text{Constant}$

**Work**

$$d\vec{w} = \vec{F} \cdot d\vec{r}$$

$dw = \text{increment of work}$

$$= m\ddot{x} \cdot \dot{x} \, dt = m \frac{d\dot{x}}{dt} \, \dot{x} \, dt = m \dot{x} \cdot d\ddot{x} = \frac{1}{2} m \dot{x} \ddot{x}$$

$$= \frac{1}{2} m \dot{x}^2 = \frac{1}{2} m v^2 = k \cdot \omega \text{ function}$$

$$= T$$

$$d\vec{w} = dT$$

$$\int_{T_1}^{T_2} \vec{F} \cdot d\vec{r} = \int_{T_1}^{T_2} dT = T_2 - T_1$$
Conservative force: The work done depends only on the initial position \( x_1 \) and the final position \( x_2 \) (not on path from \( x_1 \) to \( x_2 \))

\[
\int_{x_1}^{x_2} F \cdot dx = \int_{x_1}^{x_2} \rho \cdot dx
\]

\( \rho = \int_{x} \epsilon \cdot dx = 0 \)

\( V(x) = \int_{x}^{R_{\text{ref}}} F_{\epsilon} \cdot dx \)  \( R_{\text{ref}} \rightarrow \) reference position vector

\( F_{\text{nc}} \rightarrow \) Non-conservative force

\( T_2 - T_1 = -(V_2 - V_1) + \int_{x_1}^{x_2} F_{\text{nc}} \cdot dx \)

\( E = T + V \)

\( E_2 - E_1 \)

\( F_{\text{nc}} \cdot dx = dE \)

\( F_{\text{nc}} \cdot dx = \epsilon \cdot E = \text{power} \)

If \( F_{\text{nc}} = 0 \), \( E = 0 \) \( \omega \), \( E = \text{const.} \)

Vibrating Systems:

\( \rightarrow \) assembly of individual Components.

Components \( \rightarrow \) 3 Classes

(1) Component forces \( \propto \) displacements
(2) " \( \propto \) velocities
(3) " \( \propto \) accelerations.

(4) Components that store & release p.E
(5) " \( \) dissipate Energy.
(6) " \( \) store & release K.E
Helical Spring

All elastic component store p.E as displacement increase.
 Release p.E as decrease

Springs assumed to be Massless.

\[ F_s = \text{Tensile force} \]
\[ \delta = \text{elongation} = x_2 - x_1 \]

For linear range,

\[ F_s \propto \delta \]
\[ F_s = k \delta = k(x_2 - x_1) \]

\[ k = \text{Spring constant or Stiffness constant} \]

\[ k \to N/m \]

\( F_s \) are elastic forces, tend to return the spring to undeformed condition. 
So called restoring force.

Springs forces are Conservative,

\[ V(\delta) = \int_0^\delta F_s \, d\delta = \int_0^\delta (-k \delta) \, d\delta = -\frac{1}{2} k \delta^2 \]

Nonlinear

\( F_s \) increases at a slower rate than \( \delta \) \( \Rightarrow \) softening springs
\( F_s \) decrees faster \( \Rightarrow \) stiffening springs

Viscous damper or dashpot

Force \( \leftrightarrow \) Velocity

Viscous damper is Massless.

\[ \frac{c}{F_d} = \frac{x_1}{x_2 - x_1} \]

\[ F_d = c \delta = c(x_2 - x_1) \]

\[ E = (-c \delta) \delta = -c \delta^2 \]

\( F_d \to \text{Smooth Curve} \]

\( \delta = x_2 - x_1 \)

\( c = \text{Coefficient of Viscous damping} \)

\( F_d \to \text{Non-conservative, depends on velocity, not on position} \]

\( E = (-c \delta) \delta = -c \delta^2 \) - we see it shows it dissipate energy.
Rigid mass in translation
motion in x-direction only

\[ F_m = M \ddot{x} \]

\[ M = \text{Mass} = \text{kg} \]

\[ F_m \propto \text{acceleration} \]

\[ T = \frac{1}{2} M \dot{x}^2 \quad \text{Mass, store K.E as velocity increase.} \]

\[ \text{Mass, release K.E as velocity decrease.} \]

\( k, c, m \) are called parameters of a system.

Do not require spatial variables to describe their location
and located at discrete points (lumped or discrete parameters).

Similar type of components relate torques to rotational motions

For torsional spring, \( M_s = k_T (\dot{\theta}_2 - \dot{\theta}_1) \)

\[ k_T \Rightarrow \frac{N \text{m}}{\text{rad}} \]

\[ \text{Torque} \rightarrow \text{Torsional spring constant} \]

For torsional viscous damper, \( M_d = C_T (\dot{\theta}_2 - \dot{\theta}_1) \)

\[ C_T \Rightarrow \frac{N \text{m s}}{\text{rad}} \]

\[ \text{Torque} \rightarrow \text{Torsional coefficient of viscous damper} \]

For fixed mass, \( M_0 = \frac{I_0}{J_0} \ddot{\theta} \)

\[ J_0 \Rightarrow \text{kg m}^2 \]

\[ \text{Torque about fixed point 0} \rightarrow \text{acceleration of rigid body about 0} \]

\[ \text{Mass moment of inertia about 0} \]

Equivalent springs, dampers, masses.

Springs in parallel and series \((k_1, k_2, \ldots, k_n)\) (for linear systems)

\[ F_3 = k_1 (x_2 - x_1) \]

\[ F_3 = k_2 (x_2 - x_1) \]

\[ F_3 = F_{S_1} + F_{S_2} \]

\[ F_3 = k_{eq} (x_2 - x_1) \]

\[ k_{eq} = \frac{1}{k_1 + \frac{1}{k_2}} \]

For \( n \) springs, \( k_{eq} = \sum_{i=1}^{n} \left( \frac{1}{k_i} \right)^{-1} \)
For torsional springs \( \Rightarrow \) Similar.
For dampers \( \Rightarrow \) Similar.

Sometimes, distributed elastic components can be treated as equivalent discrete springs.

\[
U(x) = F(x) = \int_0^x \frac{dS}{EA(S)}
\]

\[
U(L) = \int_0^L \frac{dS}{EA(S)} = \frac{F}{E}
\]

\[
\text{Discrete spring:} \quad \frac{F}{S} = \text{Spring Constant}
\]

\[
\text{Axial stiffness:} \quad \frac{F}{\delta} = \text{Axial Stiffness}
\]

\[
\text{Bending Stiffness:} \quad \frac{F}{\delta} = \int_0^L \frac{dS}{EA(S)}
\]

\[
\text{Torque Moment of Inertia:} \quad EI(x) \frac{d^2w(x)}{dx^2} = M(x)
\]

\[
W(x) \Rightarrow \text{Translational Displacement}
\]

\[
EI(x) \text{ flexural rigidity}
\]

\[
\text{Beam theory:} \quad W(x) \rightarrow \text{Trans. displacement}
\]

\[
GJ(x) \rightarrow \text{replaces EA(x)}
\]

\[
\text{Stiffness} \Rightarrow \text{Resistance to deformation by an elastic body.}
\]

\[
\text{For torsion of rod, } \text{keq} = \frac{M}{\delta} = \frac{GJ}{L}
\]

\[
W(0) = 0, \quad W(L) = \text{value, } \text{w to D.F., we get}
\]

\[
\text{Key for any beam condition}
\]

\[
\text{Modelling of Mechanical Systems:}
\]

\[
\text{2 types - (1) Lumped parameter } \Rightarrow \text{discrete.}
\]

\[
\text{(2) Distributed parameter } \Rightarrow 
\]

\[
M \Rightarrow \text{mass}
\]

\[
K \Rightarrow \text{stiffness}
\]

\[
W \Rightarrow \text{velocity}
\]

\[
\text{If nonuniformly given,}
\]

\[
\text{Rubber damper } \Rightarrow \text{bilinear viscous elasically.}
\]

\[
\text{Sponges & dampers no def.}
\]
System: D.E. of motion:

- Excitation → Free vibrations
- Applied/ external excitations or forces or Moments (Impressed forces) → Forced vibrations

\[ K_1 + K_2 = k \]
\[ C_1 + c_2 = C \]

Displacement \( y(t) \) caused by \( -k y(t) - c y(t) = M g \), \( w = mg \).

\[ u, M y(t) + C y(t) + k y(t) = -M g \rightarrow \text{Non homogeneous eqn.} \]

\( x(t) \rightarrow \text{displacement from stable equilibrium position} \)
\( y(t) \rightarrow 1, \text{from unstrained spring position} \)

\[ \delta_{st} = \frac{w}{k} = \frac{mg}{k} \]
\[ y(t) = x(t) - \delta_{st} \]
\[ y''(t) = x''(t) \]

\( \delta_{st} = \text{constant} \)

\[ M x''(t) + C x(t) + k x(t) = 0. \rightarrow \text{is homogeneous.} \]

Weight did not disappear

Weight is balanced all time \( M g = k \delta_{st} \).

\( \text{by constant force of spring} \)

Automobile Car:

- Body chassis \( \rightarrow \text{capable of elastic deformations} \)
- \( L \rightarrow \text{assumed as rigid slab} \)
- Mass of slab is supported by primary suspension systems at each of 4 wheels. (Soft Springs), a hydraulic shock absorber (Viscous damper).

Vertical translation \( x(t) \) of the body

Notations: \( y(t) \) & \( \theta(t) \) of the body about axes \( y \) & \( z \) respectively & the vertical translations \( x(t) \).

System parameters:

- \( M_b \rightarrow \text{mass of the body} \)
- \( I_y, I_z \rightarrow \text{of the body axes y & z} \)
- \( C_0 \rightarrow \text{Coefficient of viscous damping of the suspension system} \)
- \( k_s \rightarrow \text{Spring constant} \)
- \( M_0 \rightarrow \text{Vehicle Masses} \)
- \( k_s \rightarrow \text{Spring constant at time} \)

\( \text{(i = 1, 2, 3, 4)} \)
For torsional springs $\rightarrow$ Similar.
For dampers $\rightarrow$ Similar.

Sometimes, distributed elastic components can be treated as equivalent discrete springs.

$$u(x) \rightarrow u(L) = \delta$$

(Torsion force)

$$k_{eq} = \frac{F}{\delta}$$

$E = $ Young's modulus

$A = $ area of cross section

$$EA(x) \frac{du(x)}{dx} = F(x), \ 0 < x < L$$

Axial stiffness

$$u(x) = F \int_{0}^{x} \frac{d\delta}{EA(\delta)}$$

$$u(L) = \delta = F \int_{0}^{L} \frac{d\delta}{EA(\delta)}$$

$$k_{eq} = \frac{EA}{L}$$

$$k_{eq} = \frac{F}{\delta} = \left[ \int_{0}^{L} \frac{d\delta}{EA(\delta)} \right]^{-1}$$

$$k_{i} = \frac{EA}{L_{i}}, \quad i = 1, 2, 3.$$  

For torsion of rod, $k_{eq} = \frac{M}{\delta} = \frac{GT}{L}$

$$E I(x) \frac{d^{2}w(x)}{dx^{2}} = M(x), \ 0 < x < L$$

(Bending moment)

Beam theory $\rightarrow$ $W(x) \rightarrow$ Trans. displacement.

$$E I(x) \rightarrow$$ flexural rigidity.

$I(x)$ $\rightarrow$ Cross. Sectional area moment of inertia.

Nominal moment of inertia of cross sectional area.

Put the $x = 0, \ L$,

$W(x) = 0, \ W(L) =$ value, $m$ to D.F, we get

$$\frac{dW}{dx}$$ values

Modeling of Mechanical Systems:

1) types $\rightarrow$ 1) Lumped parameter $\rightarrow$ discrete.

2) distributed parameter $\rightarrow$ nonuniformly.

No elastic deformations $\rightarrow$ nonuniformly.

Cloth, cloth, spread uniformly, and symmetric $\rightarrow$ springs and dampers.

Rod, bar models $\rightarrow$ behave as viscoelasticity.

Springs and dampers $\rightarrow$ UEF.
Slender elastic body → Bending about two transverse axes
Vibration takes place in one plane only → plane of the missile trajectory
Discrete model of missile, made by dividing the mass into 'r' lumps of mass, $M_i$ ($i = 1, 2, \ldots, n$) connected by massless segments of length $\Delta x_i$ and bending stiffness $EI_i$ ($i = 1, 2, 3, \ldots, n-1$)
Transverse displacement $w_i(t)$ of mass $M_i$ modeled as a distributed parameter system, in the form of a beam of length $L$, free at both ends, undergoing bending vibration $w(x,t)$ in the transverse direction.

System parameters: $M(x)$, $EI(x)$

We have $M \dddot{x}(t) + c \ddot{x}(t) + kx(t) = 0$

Consider an imbalance in the system, as shown.

To derive eqn of motion, a FBD to be considered.
One for $M - m'$ and one for $M'$

$x(t)$ measured from static equilibrium position (we can ignore $Mg$)

$-F_v - kx - c\ddot{x} = (M - m)x''$

we know, $F_v = m \frac{d^2}{dt^2} (x + \varepsilon \sin \omega t) = m (\dddot{x} - \varepsilon \omega^2 \sin \omega t)$

Substitute, $M \dddot{x} + c \ddot{x} + kx = F = m \varepsilon \omega^2 \sin \omega t = F(t)$
No of unknowns in D.E. of motion = N0 of degree of freedom of system.

Nature of excitations:

Initial excitation → eqns are homogeneous
Applied forces → eqns are Non-homogeneous.

Initial excitation consists of Initial displacement, initial velocities, potential energy (P.E) & kinetic energy (K.E) to a system.
Response of initial excitation → Free Vibrations.
If the system is Conservative, Total energy remains constant, motion is persisting ad infinitum.

If there is damping in the system, energy is dissipated, total energy decay continuously to zero, at which point motion stops.
All practical systems are damped, but still considered to be conservative as they dissipate energy very slowly. Still all motions caused by initial excitations come to rest eventually → Initial excitation referred to as transient excitations.

If there is a large variety of applied forces, D.E. of motion is Non-homogeneous.

A particular class of applied forces are harmonic excitations → Forces proportional to the trigonometric functions Sin wt, Cos wt, or Combination of Sin & Cosine.

Real life systems are harmonic

Nature of harmonic functions:

Combination of Sin wt & Cos φ:

\[ F(t) = A_1 \sin wt + A_2 \cos φt = A \cos (wt - \phi) \]

\[ w \rightarrow \text{frequency of harmonic function} \quad \text{rad/sec} \]

\[ A = \sqrt{A_1^2 + A_2^2} \quad \rightarrow \text{amplitude} \]

\[ \phi = \tan^{-1} \frac{A_2}{A_1} \quad \rightarrow \text{phase angle} \]

Geometrically, as the vertical projection of a vector A rotating with the angular velocity \( w \), the angle \( wt - \phi \) between A & the vertical axis increases linearly with time. So that the vertical projection varies harmonically with time.
The function repeats itself every time interval $T$ (seconds) \( \rightarrow \) period
\[ T = \frac{2\pi}{\omega} \]
There is time interval \( = \frac{\pi}{\omega} \) between \( F(0) \) and 1st peak \( \rightarrow \) phase difference. and is ignored at most of the times if damping is zero. For a damped system, there is phase angle between excitation & response, and not affected by \( y \) vs. \( u \). The origin location at \( t = 0 \) of the time axis has no meaning.
For a harmonic function \( \rightarrow \) two things are necessary: 
- Amplitude
- Frequency
- "no role for time" \( \omega \), harmonic excitations have the same characteristics for all times, \(-\infty < t < \infty\), for which reasons they are known as steady-state excitations. \( \rightarrow \) different form.
- Transient excitations, where \( t = 0 \), is important, \( u \) initial position & velocity one relevant.

In dealing with harmonic excitations, it is convenient to use exponential form than trigonometric form.
\[
e^{i\omega t} = 1 + i\omega t + \frac{1}{2!} (i\omega t)^2 + \frac{1}{3!} (i\omega t)^3 + \frac{1}{4!} (i\omega t)^4 + \cdots
\]
\[
= 1 - \frac{1}{2!} (\omega t)^2 + \frac{1}{4!} (\omega t)^4 + \cdots + i \left[ -\frac{1}{3!} (\omega t)^3 + \frac{1}{5!} (\omega t)^5 + \cdots \right]
\]
\[
= \cos \omega t + i \sin \omega t , \quad i = \sqrt{-1}
\]
The exponential form \( e^{i\omega t} \) is represented in the complex plane as a vector of unit magnitude & magnitude angle \( \omega t \) with respect to the real axis.

- Projection of the vector on the real axis is \( \cos \omega t \)
- Projection of the vector on the imaginary axis is \( \sin \omega t \)
As \( t \) increases, the vector rotates in the complex plane with the angular velocity \( \omega \), causing two projections to vary harmonically with time.
\[
\text{Re } e^{i\omega t} = \cos \omega t \quad \text{Im } e^{i\omega t} = \sin \omega t
\]
If \( y = 0 \), \( F(t) = \text{Re } A e^{i\omega t} \)

The advantage in exponential form is to represent in simple form. e.g., if excitation is \( \propto \cos \omega t \) \( \rightarrow \) we take real part of response.
If excitation is \( \propto \sin \omega t \) \( \rightarrow \) we take imaginary part of response.
Harmonic functions repeat themselves every time interval $T \rightarrow$ class of periodic functions $F(t)$, $T$ is period.

In periodic functions, $t$ is meaningless.

Periodic excitations represent a more general class of steady-state excitations. Hence, as harmonic functions are periodic, periodic functions are not necessarily harmonic.

Periodic functions can be expressed as linear combinations of harmonic functions known as Fourier series.

- It can be expressed as a real or complex form of $F_S$.

Frequency of each harmonic function in $F_S$ is an integer multiple of the lowest frequency $\rightarrow$ called fundamental frequency.

The plot between the amplitude of each of the constituent harmonic functions in the $F_S$ as a function of frequency $\rightarrow$ It gives the degree of participation of each of these harmonic functions in $F(t)$ $\rightarrow$ diagram called Frequency Spectrum. $\rightarrow$ It represents a frequency domain description of a periodic function. $\rightarrow$ Frequency domain description of $F(t)$ is more useful as it contains the plot of the frequency spectrum of a periodic function at the individual frequencies than a continuous plot of a particular frequency in $F(t)$ over time domain.

- Discrete frequency spectrum.

Non periodic excitation: $\rightarrow$ consists of large variety of forces.

- Impulse fun, Step fun.

$\rightarrow$ Represent arbitrary excitations

In harmonic, periodic, non periodic excitations, the advance value of force at a time $t$ is known $\rightarrow$ hence they are deterministic.

Non deterministic are called Random vibration $\rightarrow$ earthquake.

Many Random excitations show statistical regularity $u$, description in terms of certain averages, such as mean value, the mean square value.
A frequency domain description is more useful than time domain description for random excitation and responses. Random responses furnished $F(t)$ can be decomposed into harmonic components. The plot of each harmonic function at every frequency $->$ called continuous frequency spectrum.

The **Superposition Principle**:

Two forces $F_1(t) + F_2(t)$ gives responses $x_1(t)$ & $x_2(t)$

$$F(t) \rightarrow \text{System} \rightarrow x(t)$$

For $F(t) = C_1 F_1(t) + C_2 F_2(t)$ $C_1$ & $C_2$ constants.

If $x(t) = C_1 x_1(t) + C_2 x_2(t) \rightarrow \text{System is Linear}$.

If $x(t) \neq C_1 x_1(t) + C_2 x_2(t) \rightarrow \text{System is Nonlinear}$.

$$M \frac{d^2 x}{dt^2} + C \frac{dx}{dt} + kx = F$$

$$m \frac{d^2 x_1}{dt^2} + c \frac{dx_1}{dt} + kx_1 = F_1 \quad \& \quad m \frac{d^2 x_2}{dt^2} + c \frac{dx_2}{dt} + kx_2 = F_2 \rightarrow (2)$$

$$0 \times x_1 + \otimes x_2 = m \frac{d^2}{dt^2} (C_1 x_1 + C_2 x_2) + C \frac{d}{dt} (C_1 x_1 + C_2 x_2) + k (C_1 x_1 + C_2 x_2)$$

$$= C_1 F_1 + C_2 F_2 = F \rightarrow (3)$$

So System is **Nonlinear**.

If $x(t)$ is linear if dependent variable $x(t)$ and all its time derivatives appear in the eqn of motion to the $1st$ power or zero power only. It is possible to judge linearity by Chebyshev D.E.

$$m \frac{d^2 x}{dt^2} + c \frac{dx}{dt} + k (x + \varepsilon x^3) = F.$$ In this, $C_1 x_1^3 + C_2 x_2^3 \neq (C_1 x_1 + C_2 x_2)^3$.

So System is **Nonlinear**.

hence a system is **Linear** if the dependent variable $x(t)$ and all its time derivatives appear in the eqn of motion to the $1st$ power or zero power only. It is possible to judge linearity by Chebyshev D.E.

If $x(t)$ is very small, system can be regarded as **Linear**.

If $x(t)$ reaches amplitudes such that $\varepsilon x^3$ is of the order of magnitude as $x$.

System is **Nonlinear**.
periodic functions one function repeat themselves every given interval

called a period $T$.

periodic functions satisfy a relation of type,

$$f(t) = f(t + T)$$

e.g., $\sin nt$, $\cos nt$ ($n = 1, 2, \ldots$) are harmonics, periodic functions are easy to work with & possess orthogonality.

so for an arbitrary periodic function $f(t)$ can be expanded in terms of series of trigonometric functions $\rightarrow$ expansion as known as Fourier Series.

Any periodic function $x(t)$ can be represented by a series of sines and cosines that are harmonically related. If $x(t)$ is a periodic function of the period $T$, $x(t)$ is represented by the Fourier Series.

$$x(t) = \frac{a_0}{2} + a_1 \cos \omega_1 t + a_2 \cos \omega_2 t + \cdots + b_1 \sin \omega_1 t + b_2 \sin \omega_2 t + \cdots \quad (10)$$

where, $\omega_1 = \frac{2\pi}{T}$, $\omega_n = n\omega_1$.

To determine the coefficients, $a_n$ & $b_n$, we multiply both sides by $\cos \omega_n t$ or $\sin \omega_n t$ and integrate each term over the period $T$.

Thus,

$$\int_{-T/2}^{T/2} \cos \omega_n t \cos \omega_m t \, dt = \begin{cases} 0 & \text{if } m \neq n \\ \frac{T}{2} & \text{if } m = n \end{cases} \quad (2a)$$

$$\int_{-T/2}^{T/2} \sin \omega_n t \sin \omega_m t \, dt = \begin{cases} 0 & \text{if } m \neq n \\ \frac{T}{2} & \text{if } m = n \end{cases} \quad (2b)$$

$$\int_{-T/2}^{T/2} \cos \omega_n t \sin \omega_m t \, dt = \begin{cases} 0 & \text{if } m \neq n \\ 0 & \text{if } m = n \end{cases} \quad (2c)$$

all terms except one on the right side of the equation will be zero & we get,

$$a_n = \frac{2}{T} \int_{-T/2}^{T/2} x(t) \cos \omega_n t \, dt \quad (3a)$$

$$b_n = \frac{2}{T} \int_{-T/2}^{T/2} x(t) \sin \omega_n t \, dt \quad (3b)$$
The F. S. Cm also be represented in terms of the
exponential fn. Substituting,
\[ \cos \omega t = \frac{1}{2} (e^{i\omega t} + e^{-i\omega t}) \quad 4a \]
\[ \sin \omega t = \frac{1}{2i} (e^{i\omega t} - e^{-i\omega t}) \quad 4b \]

Substitute in (i), we get,
\[ x(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[ \frac{1}{2} (a_n - ib_n)e^{i\omega t} + \frac{1}{2} (a_n + ib_n)e^{-i\omega t} \right] \]
\[ = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[ C_n e^{i\omega t} + C_n^* e^{-i\omega t} \right] \quad 5a \]
\[ = \sum_{n=-\infty}^{\infty} C_n e^{i\omega nt} \quad 5c \]

where \( C_0 = \frac{1}{2} a_0 \)
\[ C_n = \frac{1}{2} (a_n - ib_n) \]

Substituting, for \( a_n \) & \( b_n \) from \( 3a, 3b \), we get:

\[ C_n = \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} x(t) (\cos \omega t - i \sin \omega t) dt \]
\[ = \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} x(t) e^{-i\omega t} \, dt \]

For computational convenience, \( x(t) = \text{Even fn} + \text{Odd fn} \)
\[ = E(t) + O(t) \]

\( E(t) \rightarrow \text{Symmetric abt Origin} \quad E(t) = E(-t) \)
\[ u, \cos \omega t = \cos (-\omega t) \]

\( O(t) \rightarrow \text{NonSymmetric abt origin, O(t) = -O(t)} \)
\[ u, \sin \omega t = -\sin (-\omega t) \]

\[ u, \int_{-\frac{T}{2}}^{\frac{T}{2}} E(t) \sin \omega t \, dt = 0 \]
\[ \int_{-\frac{T}{2}}^{\frac{T}{2}} O(t) \cos \omega t \, dt = 0 \]
The plot of F.S. Coefficient $a_n$ vs Frequency $W_n$.
We get series of discrete lines called the Fourier Spectrums.

$$|2C_n| = \sqrt{a_n^2 + b_n^2}$$

Phase $\phi_n = \tan^{-1} \frac{b_n}{a_n}$

FFT (Fast Fourier Transform) algorithm used to minimize computational time.

Convolution Integral.
Assume that the impulse response is known.

The response of a linear system with constant coefficients $\rightarrow$ expressed as Superposition of impulse responses of diff. magnitudes at diff. times $\rightarrow$ This Superposition called the Convolution Integral or Superposition Integral.

Vibration about equilibrium points:
Consider a Single-degree-of-freedom System.

$$m\ddot{y}^0 = F(y, \dot{y}) \rightarrow m = \text{Mass}$$

When $\ddot{y} = \ddot{y}_e = \text{Const.}$

$\ddot{y} = 0$ \hspace{1cm} called equilibrium points

$\dot{y} = 0$ \hspace{1cm} called equilibrium points

$F(\ddot{y}_e, 0) = 0$ after substituting $y_1$

$\rightarrow$ (2) is a polynomial eqn and No. of solutions $\leq$ degree of polynomial.

If $\nabla F(y_e, 0)$ is linear $\rightarrow$ only one solution.

If $F(y_e, 0)$ is transcendental $\rightarrow$ infinitely many solutions.

If $y_e = 0$ $\rightarrow$ is an equilibrium point but is trivial.

1. If a system disturbed from an equilibrium point returns to the same equilibrium point $\rightarrow$ motion is asymptotically stable.
2. If a system disturbed from an equilibrium point oscillate about the same equilibrium point without exhibiting any secular trend, y, the system neither returns to the equilibrium point nor moves away from it with time, then the motion is marginally stable.

We have, \( y(t) = y_e + x(t) \) \( x(t) \rightarrow \) relatively small displacement from equilibrium.

\[
\begin{align*}
\dot{y}(t) &= x(t), \\
\dot{x}(t) &= \ddot{x}(t). \\
\end{align*}
\]

Expand \( F(y, \dot{y}) \) in Taylor series about an equilibrium point \( y_e \).

\[
F(y, \dot{y}) = F(y_e, 0) + \frac{\partial F(y, \dot{y})}{\partial y} \bigg|_{y=y_e, \dot{y}=0} x + \frac{\partial F(y, \dot{y})}{\partial \dot{y}} \bigg|_{y=y_e, \dot{y}=0} \dot{x} + o(x^2).
\]

Terms of higher order and higher order in \( x, \dot{x}, \ddot{x} \), nonlinear terms.

Let \[
\frac{1}{m} \left. \frac{\partial F(y, \dot{y})}{\partial y} \right|_{y=y_e, \dot{y}=0} = -b
\]
\[
\frac{1}{m} \left. \frac{\partial F(y, \dot{y})}{\partial \dot{y}} \right|_{y=y_e, \dot{y}=0} = -a.
\]

We get, \( \dddot{x} + ax + bx = 0 \) \( \rightarrow \) linearised eqn of motion about equilibrium at small motions.

At \( x(t) = Ae^{st} \), \( A \rightarrow \) ln consequential amplitude, \( S \rightarrow \) constant of exponent.

So behaviour of system is decided by \( 'S' \).

\[
\begin{align*}
x(t) &= Ae^{st}, \\
\dddot{x}(t) &= Ase^{st}.
\end{align*}
\]

Substitute, \( Ase^{st} + x + bAe^{st} = 0 \)

\[
\begin{align*}
s^3 + as + b & = 0 \rightarrow \text{characteristic eqn,} \\
S_1 &= \frac{-a}{2} + \sqrt{\left(\frac{a}{2}\right)^2 - b}.
\end{align*}
\]

\( S_2 \)
Solution 3, \( x(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t} \)

- \( s_1 \) and \( s_2 \) are in general complex.
- \( s_1, s_2 \) can be real, pure imaginary, complex (real + imaginary).

For \( x(t) \) to be real:
- \( s_1, s_2 \) must be imaginary or complex.
- \( s_1 \) and \( s_2 \) are complex conjugates.
- So \( A_1 \) and \( A_2 \).

When \( s_1 \) and \( s_2 \) are real and \( s_1 = -s_2 \), solution approaches asymptotically zero.

- \( s_1 \) and \( s_2 \) are complex, \( \rightarrow \) magnitude of \( x(t) \) controlled by real part of \( s_1 \) and \( s_2 \).
- \( s_1 \) and \( s_2 \) are complex, with \( -\text{re } s_1 \) real part.

- \( \rightarrow \) solution approaches zero in an oscillatory fashion as \( t \to \infty \).

- \( s_1 \) and \( s_2 \) are imaginary \( \rightarrow \) motion is harmonic only, merely stable.

- \( s_1 \) and \( s_2 \) are real and positive \( \rightarrow \) solution diverges, unstable motion.

- \( a > 0, b > 0 \) \( \rightarrow \) asymptotically stable motion 1.
- \( a < 0, b > 0 \) \( \rightarrow \) asymptotically unstable motion 1.
- \( a > 0, b = 0 \) \( \rightarrow \) periodic decay.
- \( a = 0, b > 0 \) \( \rightarrow \) periodic decay.
- \( a < 0, b > 0 \) \( \rightarrow \) oscillatory decay.
- \( a > 0, b < 0 \) \( \rightarrow \) pure oscillatory decay.
- \( a < 0, b < 0 \) \( \rightarrow \) unstable.
A periodically decaying motion

Prime harmonic oscillation

Damped oscillation

Diverging oscillation

A periodically diverging motion
Response of a Single-degree-of-freedom Systems to Initial Excitation

Vibration of a system in response to a limited excitation. Considering all displacements, velocities, → free vibration.

→ homogeneous ordinary diff. eqn.

Undamped SDOF systems (Harmonic Oscillators)

Here, \( C = 0 \) (damping is zero).

Eqn of motion for free vibration: \( \ddot{x} + \frac{k}{m} x = 0 \)

\[ m \ddot{x}(t) + k x(t) = 0 \]

\( m \ddot{x}(t) + k x(t) = 0 \) → displacement from static equilibrium point.

\[ \frac{0}{m} \implies \ddot{x}(t) + \frac{k}{m} x(t) = 0 \implies \omega_n = \sqrt{\frac{k}{m}} \]

→ Natural frequency of the system

\[ m \ddot{x}(t) + \omega_n^2 x(t) = 0 \]

Initial conditions: \( x(0) = x_0 \) (Initial distp.

\( \dot{x}(0) = v_0 \) (Initial velocity)

\( \dot{x}(t) = A e^{st} \) be the solution to \( x(t) \)

\( \ddot{x}(t) = A s e^{st} \)

\( \dddot{x}(t) = A s^2 e^{st} \)

Subst. we get,

\[ s^2 + \omega_n^2 = 0 \] → char. eqn.

\( s = \lambda \) has 2 pure imaginary complex conjugates

\( S_1 = \pm j \omega_n \cdot \quad S_2 = -\omega_n \)

\( S = \pm j \sqrt{-\omega_n} = \pm j \omega \), \( \omega_n = \pm j \omega \)

\( x(t) = A_1 e^{j \omega t} + A_2 e^{-j \omega t} \) → general solution with \( s, \omega \)

\( A_1, A_2 \) → constants of integration & complex No.

\( A_2 \) is the complex conjugate of \( A_1 \)
Any Complex No \( A_1 = \frac{c}{2} e^{-i\phi} \) has magnitude with exponential with real Imaginary exponent. 

\[ A_2 = \frac{c}{2} e^{+i\phi} = \overline{A_1} \]

\[ e^{i\phi} + e^{-i\phi} = 2 \cos \phi \]

\[ n(t) = \frac{c}{2} \left[ e^{i(w_0 t - \phi)} + e^{-i(w_0 t - \phi)} \right] = c \cos(w_0 t - \phi) \]

Harmonic Oscillation

- **Amplitude**
- **Phase Angle**

We know, 
\[ n(0) = x_0 = c \cos \phi \]
\[ x_0(0) = V_0 = w_0 c \sin \phi \]

\[ c = \sqrt{x_0^2 + \left(\frac{V_0}{w_0}\right)^2} \]
\[ \phi = \tan^{-1} \frac{V_0}{x_0} \]

Let \( \cos(x - \beta) = \cos x \cos \beta + \sin x \sin \beta \)
\[ n(t) = c \cos(w_0 t - \phi) = c \left( \cos(w_0 t) \cos \phi + \sin(w_0 t) \sin \phi \right) \]

Once Oscillation is set, it continues indefinitely.

Dissipation or no gain of energy \( \rightarrow \) Conservative System.

**Example:** Simple pendulum, \( \ddot{\theta} + \omega_0^2 \theta = 0 \)
\[ \omega_0 = \sqrt{g/L} \]

\[ T = \omega_0 \]
L = total length of column.

velocity of liquid is zero

System is conservative

\[ p \cdot E = V = \rho g A x \cdot \frac{x}{2} + -\rho g A n \left(-\frac{x}{2}\right) = \rho g A n^2 \]

\[ k \cdot E = T = \frac{1}{2} m n^2 = \frac{1}{2} \rho g A L n^2 \]

\[ T \cdot E = V + T = \frac{1}{2} \rho g A n^2 + \frac{1}{2} \rho A L n^2 \]

\[ \frac{E}{E} = \rho A L \left( \frac{1}{2} n^2 + \frac{9}{L} n^2 \right) \]

\[ \frac{E}{E} = \rho A L \left( \frac{n^2}{2} + 2 \frac{9}{L} n^2 \right) = \rho A n \left( \frac{n^2}{2} + 2 \frac{9}{L} n^2 \right) = 0 \]

\[ n^2 + 2 \frac{9}{L} n = 0 \]

\[ \frac{n}{\sqrt{2} g} = n_0 \]

\[ L = \frac{\omega n}{\omega n} \]

\[ \omega = \sqrt{2} g \]

Viscously damped SDOF Systems:

\[ m \ddot{x}(t) + (\dot{x}(t) + kx(t)) = 0 \]

\[ \frac{0}{m} = \frac{0}{m} \]

\[ \ddot{x}(t) + \frac{c}{m} \dot{x}(t) + \frac{k}{m} x(t) = 0 \]

\[ \ddot{x}(t) + 2 \alpha w_n \dot{x}(t) + w_n^2 x(t) = 0 \]

\[ \alpha = \frac{c}{2\omega m} \]

\[ w_n = \sqrt{\frac{k}{m}} \]

\[ S = \frac{c}{2\alpha \omega m} \]

\[ w_n = \sqrt{\frac{k}{m}} \]

\[ S = \frac{c}{2\omega m} \]

\[ \omega = \sqrt{\frac{k}{m}} \]

Non-dimensional Qh

Called viscous damping factor

\[ x(t) = Ae^{\lambda t} \]

\[ s^2 + 2\alpha w_n s + w_n^2 = 0 \]

Limiting condition:

\[ x(0) = x_0 \]

\[ \dot{x}(0) = \alpha v_0 \]

\[ s^2 + 2\alpha w_n s + w_n^2 = 0 \]

Euler-Lagrange eqn.
$s_1 = -\alpha \omega_n \pm \sqrt{\alpha^2 - 1} \cdot \omega_n$


equation of motion depends on roots $s_1 \& s_2 \rightarrow \ln \text{turn depend on}

\text{value of the parameter } \alpha,

\text{here } \alpha > 0

\text{plot of locus of roots } s_1 \& s_2 \text{ as fn. of } \alpha

\text{at a given value of } \omega_n.

\text{S-plane (Complex Plane)}

\begin{align*}
\text{Aperiodic decay} \quad & \alpha = 1 \\
\text{Oscillatory decay} \quad & \alpha > 1
\end{align*}

\text{Parameter plane:}

\text{The region } 0 < \alpha < 1 \text{ is commonly referred to as underdamping.}

\text{For } \alpha = 1 \text{ roots } s_1 \& s_2 \text{ coalesce at point } -\omega_n \text{ on the real axis of } S_\text{-plane}.

\text{They correspond to the parabola } \alpha = 1 \text{ in the parameter plane.}

\begin{align*}
\text{— Critical damping} & \quad \text{— motion called aperiodic decay} \\
\text{for } \alpha > 1, \text{ both roots } s_1 \& s_2 \text{ located on the } -\text{ive real axis of the } S_\text{-plane;}
\end{align*}

\text{with } s_1 \text{ between the point } -\omega_n \text{ & origin}

\text{left of } -\omega_n

\text{as } \alpha \text{ increases, } s_1 \rightarrow 0 \text{ & region between } s_2 \rightarrow -\infty \Rightarrow \text{ aperiodic decay } \Rightarrow \text{ over damping}

\text{hence in damped free vibration, } \alpha \text{ plays a role while } \omega_n \text{ donot affect type of motion. So } s_1, s_2 = \text{fn} (\alpha) \text{ where } \omega_n = \text{const. Thus we plot for different values of } \omega_n.
\[
 y(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t} \quad \text{where} \quad A_1, A_2 \]

Initial Conditions:
\[
 y(0) = A_1 + A_2 = x_0 \\
 y'(0) = s_1 A_1 + s_2 A_2 = v_0
\]

Solving, we get:
\[
 A_1 = \frac{-s_2 x_0 + v_0}{s_1 - s_2}, \quad A_2 = \frac{s_1 x_0 - v_0}{s_1 - s_2}
\]

Thus,
\[
y(t) = \frac{-s_2 x_0 + v_0}{s_1 - s_2} e^{s_1 t} + \frac{s_1 x_0 - v_0}{s_1 - s_2} e^{s_2 t}
\]

For underdamped systems, \(0 < \delta < 1\), \(\delta_i = s_i\),

\[
 W_d = \sqrt{1 - \delta^2}, \quad \omega_d \rightarrow \text{frequency of damped vibration}
\]

We know:
\[
e^{i\omega t} - e^{i\omega_0 t} = 2\delta \sin \omega_0 t
\]

\[
e^{i\omega t} + e^{i\omega_0 t} = 2\cos \omega_0 t
\]

Thus,
\[
y(t) = e^{-\delta \omega t} \left\{ \left[ (-\delta \omega + i\omega) x_0 + v_0 \right] e^{i\omega t} + \left[ (-\delta \omega - i\omega) x_0 - v_0 \right] e^{-i\omega t} \right\}
\]

\[
= e^{-\delta \omega t} \left[ \frac{-\delta \omega x_0 + v_0}{\omega_d} \sin \omega_d t + x_0 \cos \omega_d t \right]
\]

\[
x(t) \rightarrow \text{as a fn of time}
\]

Amplitude:
\[
C = \sqrt{x_0^2 + \left( \frac{-\delta \omega x_0 + v_0}{\omega_d} \right)^2}
\]

Phase:
\[
\phi = \tan^{-1} \left( \frac{-\delta \omega x_0 + v_0}{\omega_d x_0} \right)
\]

Product: Shows that amplitude is decaying.
Time varying amplitude provides the exponentially nonuniform envelope
\[ + Ce^{-\xi \omega t} \cdot \text{modulating the harmonic fn. } \cos(\omega t - \phi) \]

Consider \( \xi > 1 \), overdamping, \( \xi_1, \xi_2 \) one real, -i.e.
\[
\begin{align*}
e^{-\xi t} - e^{\xi t} &= 2 \sinh \xi t \\
e^{\xi t} + e^{-\xi t} &= 2 \cosh \xi t
\end{align*}
\]

\[
x(t) = \frac{e^{-\xi \omega t}}{2\sqrt{\xi^2 - 1}} \cdot \omega_n \left\{ \left[ -\left( -\xi \omega_n - \sqrt{\xi^2 - 1} \cdot \omega_n \right)x_0 + v_0 \right] e^{\left( \sqrt{\frac{\xi^2}{2}} - 1 \right) \omega t} \\
+ \left[ -\left( -\xi \omega_n + \frac{\sqrt{\xi^2 - 1} \cdot \omega_n}{2} \right)x_0 - v_0 \right] e^{-\left( \sqrt{\frac{\xi^2}{2}} - 1 \right) \omega t} \right\}
\]

\[
= e^{-\xi \omega t} \cdot \left[ \frac{\xi \omega_n x_0 + v_0}{\sqrt{\xi^2 - 1}} \cdot \omega_n \sinh \left( \sqrt{\frac{\xi^2}{2}} - 1 \right) \omega t + x_0 \cosh \left( \sqrt{\frac{\xi^2}{2}} - 1 \right) \omega t \right]
\]

- Represents periodic decay.
Consider $\delta = 1$, critically damped oscillation.

$S_1 + S_2 = -\omega_n$.

Response obtained by Laplace method.

Limiting case of overdamping is critically damping, $\omega, \delta \to \infty$.

$$\delta \to 1 \quad \sinh \sqrt{\delta^2 - 1} \omega_nt = t,$$

$$\delta \to 1 \quad \cosh \sqrt{\delta^2 - 1} \omega_nt = 1.$$

Expand series for $\sinh \sqrt{\delta^2 - 1} \omega_nt$ or by 'L Hospital's Rule'.

$$x(t) = [x_0 + (\omega_n x_0 + v_0)t] e^{-\omega nt}.$$

Measurement of Damping:

Vibration measurement terminology:

1. Peak value: Indicates the maximum response of a vibrating part.
   It also places a limitation on the "Rattle Space" requirement.
2. Average value: Indicates a steady state or static value.

$$\overline{x} = \lim_{T \to 0} \frac{1}{T} \int_{-T/2}^{T/2} x(t) \, dt$$
\[ x(t) = A \sin(\omega t) \]

\[ x^2 = \lim_{T \to \infty} \frac{1}{T} \int_0^T x^2(t) \, dt \]

\[ \text{RMS} \text{ value: This is the square root of the mean square value.} \]

\[ x_{\text{rms}} = \sqrt{\langle x^2 \rangle} = \sqrt{\frac{A}{2}} = 0.707A \]

\[ \text{Decibel (dB): It is a unit of the relative measurement of the vibration.} \]

\[ \text{Sound: It is defined in terms of a power ratio:} \]

\[ D_b = 10 \log_{10} \left( \frac{P_1}{P_2} \right) \quad p = \text{power} \propto A^2 \]

\[ D_b = 10 \log_{10} \left( \frac{A_1}{A_2} \right)^2 = 20 \log_{10} \left[ \frac{A_1}{A_2} \right] \]

\[ A \text{ is the amplitude} \]
Octave

If 2 frequencies have ratio 2:1, the frequency span is one octave.

\[ \text{Octave} = \log_2 \left( \frac{f_{\text{max}}}{f_{\text{min}}} \right) \]

Measurement of Damping

Normally, damping factor \( \delta \) is measured rather than damping coefficient \( C \).

It uses the system's natural frequency instead of the damping constant alone.

A drop in the value of amplitude at the end of one complete cycle of vibration is called damping.

Let \( t_1 \) and \( t_2 \) be the times corresponding to the 1st and 2nd peaks.

Let \( x_1 \) and \( x_2 \) be the displacements at \( t_1 \) and \( t_2 \).

The rate of decay of oscillations measured by the logarithmic decrement is

\[ \frac{x_1}{x_2} = \frac{2\pi \sqrt{1 - \delta^2}}{\sqrt{2\pi \delta}} = e^{\frac{2\pi \delta}{\sqrt{1 - \delta^2}}} \]

Take natural log. on both sides.

\[ \delta = \frac{\ln \left( \frac{x_1}{x_2} \right)}{2\pi \delta} = \sqrt{1 - \delta^2} \]

For small damping, \( \delta \ll 1 \),

\[ \delta \approx \frac{\delta}{2\pi} \]
\[ \alpha, \beta \] (damping factor) can be determined by measuring the displacements at two different times separated by a given No. of periods.

Let \( x_j \) and \( x_{j+1} \) be the peak displacements at times \( t_j \) and \( t_{j+1} \), respectively.

We say, \( x_j = x_1 x_2 \ldots x_j \) and \( x_{j+1} = x_2 x_3 \ldots x_{j+1} \).

We get logarithmic decrement,

\[ \delta = \frac{2 \alpha \delta}{\sqrt{1-\beta^2}} = \frac{-1}{\sqrt{1-\beta^2}} \ln \frac{x_1}{x_{j+1}} \]

The measurement of two peak values \( x_j \) and \( x_{j+1} \) leads to errors, particularly for small damping.

So accuracy improved by,

\[ \ln x_j = \ln x_{j+1} - \delta (n-1), \quad j = 1, 2, \ldots, n \]

The ratio of amplitude after \( n \) cycles is \( \frac{x_j}{x_1} = \frac{x_n}{x_1} \).

So if we take any values of

\[ \log \frac{x_j}{x_1} = n \delta \]

at 50% reduction of amplitude

\[ \delta = \frac{2 \pi \delta}{n} = \frac{1}{n} \ln 2 = 0.693 \]

\[ n \delta = 0.693 \frac{2 \pi}{n} = 0.110 \]
Coulomb Damping - Dry Friction:
- It occurs when bodies slide on dry surfaces.
- Till the motion starts $\Rightarrow$ Force acting upon the body just before the motion starts is friction.

The dry friction force is not to the Surface and proportional to the force Normal to the Surface.

Consider a Mass-Spring System,

\[ \text{Normal force} = \text{Weight} \cdot W \]

Constant of proportionality $\mu_s$ --> static friction coefficient
vary between 0 to 1

Once motion is initiated, the force drops to:

\[ \mu_k W \]

where $\mu_k$ --> kinetic friction coefficient

$\mu_k < \mu_s$.

Frictional force is opposite direction to velocity.

\[ F_{friction} = \text{Const. till inertia force, spring force overcome weight} \]

\[ F_d \rightarrow \text{magnitude of dampening force} \]

\[ F_d = \mu_k W \]

\[ \text{eqn of motion, } m\ddot{x} + F_d \cdot \text{Sgn}(x) + kx = 0 \rightarrow (1) \]

Sgn denotes $\text{Signum function}$ or the $\text{Sign of}$.

by sign having the value +1 if argument $x$ is +ve

-1, if $x$ is -ve

Mathematically, $\text{Sgn}(x) = \frac{x}{|x|} \rightarrow (2)$

\( (1) \) is Nonlinear, but can be Split into two linear eqns,

\[ m\ddot{x} + kx = -F_d \text{ for } x > 0 \]

\[ m\ddot{x} + kx = F_d \text{ for } x < 0 \rightarrow (3) \]

Represent forced vib.

$\ddot{x}$ damping is passive
Solution for \( t \) + for \( \omega t + \) time interval at a time...

m's' displacement from rest, \( x(t) = x_0 + \dot{x}_0 t + \frac{1}{2} \ddot{x}_0 t^2 \) is the sufficiently large...

\( x_0 \rightarrow \text{limit displacement} \) is large so that the restoring force in spring exceeds the Fstatic fmem.

\[ x(t) + Wn^2 x = \frac{1}{m} \]

\( x(t) = x_0 \cos \omega t + \frac{1}{\omega} \sin \omega t \) for \( 0 \leq t \leq t_1 \)

\( t_1 \rightarrow \text{time at which velocity} \rightarrow \text{zero} \)

\( \dot{x}(t) = -\omega_n (x_0 - \frac{1}{\omega} \sin \omega t) \sin \omega t \)

\( \ddot{x}(t) = 0 \) at \( t_1 = \frac{t_1}{\omega} \)

\( x(t) = -x \)

If \( x(t) \) is large enough to overcome Fstatic, then M moves left to right velocity becomes the...

\( \dot{x}(t) \rightarrow + \text{ time} \)

motion satisfy \( x(t) + Wn^2 x = -Wn^2 f \) is the eqn. to satisfy

\( x(t) \) subject to initial conditions,

\( x(t) = -(x_0 - 2f \dot{x}) \)

\( \ddot{x}(t) = 0 \)

apply to (6) \( x(t) = (x_0 - 3f \dot{x}) \cos \omega t - f \dot{x} \rightarrow (6) \)

Compare Soln's at time \( t_1 \) and at time \( t_2 \),

\( x(t) = \) the amplitude is Smaller by \( -2f \dot{x} \) average response is by \( -f \dot{x} \)

Solution is valid as \( t_1 \leq t \leq t_2 \), \( t_2 = t + T \) next value of \( t \) at which the velocity reduces to zero.

\( t = \frac{2\pi}{\omega} \) \( \text{at which the velocity ready to reverse direction} \).
The displacement at \( t = t_2 \) is

\[
x(t_2) = x_0 - 4f_d
\]

The procedure is repeated for \( t > t_2 \)

The average value of the \( x(t) \) alternate between \( 
\frac{f_d}{2} \) and \( -\frac{f_d}{2} \).

At the end of each half-cycle, the displacement magnitude is reduced by

\[
2f_d = \frac{2f_d}{k}
\]

- \( \kappa \), Coloumb damping, this decay is linear v. t.
- In viscous damping, the decay is exponential v. t.

The motion stops suddenly when the displacement at the end of a given half-cycle is not sufficiently large for the restoring force in Spring to overcome the static friction. This occurs when the amplitude in the harmonic component is smaller than \( 2f_d \).

Let \( n \) be the No. of half-cycles just prior to the cessation of motion; \( n \) is the smallest integer satisfying the inequality:

\[
x_0 - (2n - 1)f_d < \left( 1 + \frac{\mu_s}{Mk} \right)f_d.
\]

**Energy method.** In a conservative system \( T + U \) is const. \( u \), \( E = 0 \)

\[
E = T + U = p + E
\]

- principle of energy conservation.

\( T \rightarrow \) Stored K.E.; in terms of velocity, Plastihilt, gravity

\( U \rightarrow \) Stored P.E., In terms of strain energy, work done, etc.

\[
\frac{d}{dt}(T + U) = 0 \quad \text{at time } t_1, \qquad T_1 = U_2
\]

Let \( t_1 = 0 \), initial condition. \( \Rightarrow T_1 + 0 = 0 + U_2 \)

\( t_2 = \max \) displacement for proceeding. So \( \cdots \) leads to natural frequency.
Determine the $W_n$ for the system shown.

Assume harmonic vibration, $\theta \rightarrow$ amplitude.

$$T_{max} = \frac{1}{2} J \dot{\theta}^2 + \frac{1}{2} m (x_0 \dot{\theta})^2$$

$$U_{max} = \frac{1}{2} k (x_2 \theta)^2$$

(1) $= (2)$

$$W_n = \sqrt{\frac{K x_2}{J + m a^2}}$$

cylinder weight $w$  Cylindrical Surface $R$

Resides $R$

No slipping $\dot{R} = \dot{\theta}$

For cylinder $\rightarrow$ both translation & Rotation

Translational velocity at the centre of cylinder $\frac{\dot{R}}{R}$

Rotational $\frac{\theta}{(\theta - \theta)} = \left(\frac{R}{R} - 1\right) \dot{\theta}$

$$K \cdot C = \frac{1}{4} = \frac{1}{2} \frac{w}{3} \left[(R-R) \frac{a^2}{2} + \frac{1}{2} \frac{w}{3} \left[(R-R) \theta^2\right]^{\frac{1}{2}}

= \frac{3}{9} \frac{w}{(R-R) \frac{a^2}{2}}$$

$$C = U \text{ at lowest position} = \frac{W (R-R) (1 - \cos \theta)}{W}$$

$$K \cdot E = P \cdot C$$

$$\frac{3}{4} \left(\frac{a^2}{2}\right) (R-R)^2 = \frac{w}{(R-R) (1 - \cos \theta)}$$

Solving, $W_n = \sqrt{\frac{2w}{3(R-R)}}$.
Rayleigh method - effective mass. Energy method used for multmass systems. Or for distributed mass → Provided the motion of every point in the system is known.

The \( k \cdot E = \frac{1}{2} M_{\text{eff}} \dot{x}^2 \). \( M_{\text{eff}} \) → effective mass.

If stiffness at that point is known, the natural frequency can be calculated as:

\[ \omega_n = \sqrt{\frac{k}{M_{\text{eff}}}}. \]

Principle of Virtual Work: (Bernouilli method) states:

D'Alembert method for dynamics, inertia force introduced.

L'Hospital's Rule:

If we have one of the following cases,

\[ \lim_{x \to a} \frac{f(x)}{g(x)} = \frac{0}{0} \quad \text{or} \quad \lim_{x \to a} \frac{f(x)}{g(x)} = \pm \infty \]

we have,

\[ \lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)} \]

Ex: \( \ln x = \ln 1 \):

\[ \lim_{x \to 0^+} \frac{\ln x}{x} = \lim_{x \to 0^+} \frac{-x}{\ln x} = -\infty \] (form.

So by L'Hospital Rule, \( \lim_{x \to 0^+} \frac{1}{x^n} = \lim_{x \to 0^+} (-x) = 0 \).

Energy dissipation:

Damped systems dissipate energy. So they are nonconservative systems.

Viscous damping, Coulomb damping:

Lin. \( \dot{x} \) velocity \( \Rightarrow \) Nonlin. \( \dot{x} \) velocity.

Energy dissipated in real springs are due to internal friction.

Damping due to external friction does not depend on velocity.

The response of a Mass-damper-spring system subject to harmonic excitation.

The three most common types of damping are:

Hysteresis damping, known as solid damping or structural damping, is the result of internal friction in the material of the body during oscillatory motion. When an elastic body deforms during its oscillatory motion, frictional forces develop inside the body due to the friction between internal planes that slip & slide during deformation. This type of resistance is independent of the frequency of vibration but approximately \( \propto \) to the amplitude of the deformed elastic body.

Consider a Spring-Mass System in SDOF and plot of Force vs displacement as shown above.

The area with in the hysteresis loop \( \Delta U \) = amount of energy spent in the spring constant transformed to heat. \( \Delta U \) per cycle of motion:

\[
\Delta U = k \frac{\pi}{4} C_0 y^2
\]

\( y \) = displacement amplitude of vib, \( m \) = mass, \( k \) = spring force = ky, \( C_0 \) = damping constant of the material (solid).

General eqn:

\[
m \ddot{x} + \frac{\partial}{\partial \dot{x}} (\sigma, \dot{x}) = p
\]

\( \frac{\partial}{\partial \dot{x}} (\sigma, \dot{x}) \) have two parts:

\[
f_s = k \dot{x}
\]

average stiffness

\[
f_D = \frac{\pi k}{2}
\]

\( n = \text{const} \)

\( \nu = \text{frequency of vib} \)

Ex: Rubber type Materials.

The effects of hysteresis damping could be taken into consideration by determining an equivalent viscous damping ratio \( \frac{\nu}{\nu} \) and equivalent viscous damping factor \( C_e \) so the eqn is

\[
y(t) = e^{-\nu \omega t} \left[ A_1 \cos \left( \omega \sqrt{1 - \frac{\nu^2}{\nu^2}} t \right) + A_2 \sin \left( \omega \sqrt{1 - \frac{\nu^2}{\nu^2}} t \right) \right]
\]

\( A_1 \) & \( A_2 \) obtained from initial condition.
The damped frequency of vibration $W_d$ for hysteretic damping

$$W_d = \frac{\omega}{\sqrt{1 - \frac{\delta^2}{\omega^2}}} \quad \omega = \text{free undamped freq.}$$

at time $t = 0$, initial displacement, $x_0$,

1st velocity $v_0$

$$A_1 = x_0$$

$$A_2 = \frac{v_0 + \delta \omega x_0}{W_d}$$

$$x(t) = e^{-\delta \omega t} \left[ x_0 \cos \omega_d t + \frac{v_0 + \delta \omega x_0}{W_d} \sin \omega_d t \right]$$

$\delta$ and $\omega$ determined from consecutive peaks.

Apply energy principles,

Energy loss per quarter cycle.

$$= \frac{k}{\pi} C_0 \frac{y^2}{4}$$

for half cycle, between $A$ & $B$

$$\frac{k x_1^2}{2} - \frac{k \bar{C} C_0 x_1^2}{4} - \frac{k \bar{C} C_0 x_2^2}{4} = \frac{k x_2^2}{2}$$

$$\frac{k}{2} \left[ x_1^2 - x_2^2 \right] = \frac{k \bar{C} C_0}{4} \left[ x_1^2 + x_2^2 \right]$$

$$x_1^2 - x_2^2 = \frac{\pi}{2} C_0 \left[ x_1^2 + x_2^2 \right]$$
Phase Space

The displacement and velocity are functions of \( t \) and there is no phase lag. A phase space is a parametric graph of \( v(t) \) plotted as a function of \( x(t) \) with changing variable being time.

Undamped Oscillator

\[
\frac{d^2x}{dt^2} + \omega_0^2 x = 0
\]

Damped Oscillator

\[
x(t) = e^{-\alpha t} \left( A \cos(\omega t) + B \sin(\omega t) \right)
\]