Steady state harmonic excitations differ from all other types. The behaviour of the response to such excitations can be extracted by frequency domain rather than time domain techniques. 

1. Frequency response plots.
2. Showing Amplitude and phase angle of the response vs. Frequency.

Rotating Components gives 

Harmonic excitations

Periodic excitations are also steady state. Harmonic functions are periodic.

But all periodic function are not necessarily harmonic.

Periodic functions represented by infinite series of harmonic funcions.

Frequency domain is good for harmonic excitations.

Response of SDOF Systems to Harmonic Excitations:

We know, 

\[ M \ddot{x}(t) + C \dot{x}(t) + k x(t) = F(t) \]

\( M = \text{mass} \)
\( C = \text{coefficient of viscous damping} \)
\( k = \text{Spring constant} \)

\( F(t) = \text{Force}; \ x(t) = \text{Displacement} \)

at \( x(0) = x_0 \)
\( \dot{x}(0) = v_0 \)

\text{Initial excitations (We already seen)}

Treated by principle of Superposition.

\( F(t) \rightarrow \text{is a harmonic force} \)

\( F(t) = k f(t) = k A \cos(\omega t) \) where \( f(t) = A \cos(\omega t) \)

W → excitation frequency or driving frequency.
\( f(t) \rightarrow \text{displacement} \)
\( A \rightarrow \text{Amplitude of displacement} \)

Substitute \( 2\) in \( 1 \): 

\[ M \ddot{x}(t) + C \dot{x}(t) + k x(t) = k A \cos(\omega t) \]

\[ \frac{1}{m} \Rightarrow \ddot{x}(t) + \frac{C}{m} \dot{x}(t) + \frac{k}{m} x(t) = \frac{k}{m} A \cos(\omega t) \]

\( \omega_n = \sqrt{\frac{k}{m}} \)

\( \delta = \frac{C}{2 \omega_n m} \)
harmonic force can be steady state excitations. 
- time plays only a secondary role. 
- response is also steady state. 

So we assume, \[ x(t) = C_1 \sin \omega t + C_2 \cos \omega t. \]

Substituting in (4), we get:

\[ \omega^2 \left( C_1 \sin \omega t + C_2 \cos \omega t \right) + \omega^2 \omega_n^2 \left( C_1 \cos \omega t - C_2 \sin \omega t \right) + \omega_n^2 \left( C_1 \sin \omega t + C_2 \cos \omega t \right) = 0. \]

Simplifying, we get:

\[ \omega_n^2 A \cos \omega t. \]

Equating coefficients of \( \sin \omega t \) and \( \cos \omega t \), we get:

\( \omega_n^2 - \omega^2 \) \( C_1 \) - \( 2 \omega \omega_n C_2 = 0 \)

\( 2 \omega \omega_n C_1 + (\omega_n^2 - \omega^2) C_2 = \omega_n^2 A \).

Using Cramer's Rule:

\[ C_1 = \frac{\begin{vmatrix} 0 & -2 \omega \omega_n \\ \omega_n^2 & \omega_n^2 - \omega^2 \end{vmatrix}}{\begin{vmatrix} \omega_n^2 - \omega^2 & -2 \omega \omega_n \\ 2 \omega \omega_n & \omega_n^2 - \omega^2 \end{vmatrix}} = \frac{-2 \omega \omega_n^3 A}{(\omega_n^2 - \omega^2)^2 + (2 \omega \omega_n)^2} \]

\[ C_2 = \frac{\begin{vmatrix} \omega_n^2 - \omega^2 & 0 \\ 2 \omega \omega_n & \omega_n^2 - \omega^2 \end{vmatrix}}{\begin{vmatrix} \omega_n^2 - \omega^2 & -2 \omega \omega_n \\ 2 \omega \omega_n & \omega_n^2 - \omega^2 \end{vmatrix}} = \frac{1 - (\omega/\omega_n)^2}{(\omega_n^2 - \omega^2)^2 + (2 \omega \omega_n)^2} \]
Substitute \( x(t) \) in (5) we get:

\[
x(t) = \frac{A}{\sqrt{\left[1 - \left(\frac{w}{w_n}\right)^2\right]^2 + \left[2 \delta \frac{w}{w_n}\right]^2}} \left\{ \frac{2 \delta w}{w_n} \sin(wt) + \left[1 - \left(\frac{w}{w_n}\right)^2\right] \cos(wt) \right\}
\]

We introduce:

\[
\sin \phi = \frac{2 \delta \frac{w}{w_n}}{\sqrt{\left[1 - \left(\frac{w}{w_n}\right)^2\right]^2 + \left[2 \delta \frac{w}{w_n}\right]^2}}
\]

\[
\cos \phi = \frac{1 - \left(\frac{\omega_0}{\omega_n}\right)^2}{\sqrt{\left[1 - \left(\frac{w}{w_n}\right)^2\right]^2 + \left[2 \delta \frac{w}{w_n}\right]^2}}
\]

So harmonic response:

\[
x(t) = X \cos \left(\omega t - \phi\right)
\]

where,

\[
X = X(\omega) = \frac{A}{\sqrt{\left[1 - \left(\frac{w}{w_n}\right)^2\right]^2 + \left[2 \delta \frac{w}{w_n}\right]^2}}
\]

\[
\phi = \phi(\omega) = \tan^{-1} \left( \frac{2 \delta \frac{w}{w_n}}{1 - \left(\frac{\omega_0}{\omega_n}\right)^2} \right)
\]

plots of \( X(\omega) \) vs \( \omega \) and \( \phi(\omega) \) vs \( \omega \) yields more information than \( x(t) \) vs \( t \) Time domain and \( X(\omega) \) vs \( \omega \). Frequency response plots.

We find the nature of \( A \) (real, imaginary etc) using exponential form.

we know, \( f(t) = A \cos(\omega t) \). \( f(t) = A e^{i\omega t} = A [\cos(\omega t) + i\sin(\omega t)] \)

\[
x(t) = x(t) = \frac{A}{\sqrt{\left[1 - \left(\frac{w}{w_n}\right)^2\right]^2 + \left[2 \delta \frac{w}{w_n}\right]^2}} \left\{ \frac{2 \delta w}{w_n} \sin(wt) + \left[1 - \left(\frac{w}{w_n}\right)^2\right] \cos(wt) \right\}
\]

If the excitation \( f(t) = A \cos(\omega t) \Rightarrow \text{Response is } Re \{ x(t) \}
If the excitation \( f(t) = A \sin(\omega t) \Rightarrow \text{Response is } Re \{ x(t) \} \)
The advantage of the complex notation is, we get solution easier.

\[ u(t) = x(t) e^{iwt} \rightarrow (1) \]

Substituting (1) in (10) yields \[ z(iw) x(iw) e^{iwt} = \frac{w_n^2 A e^{iwt}}{Z(iw)} \rightarrow (12) \]

Where \[ Z(iw) = \frac{w_n^2 - w^2 + i2\xi w_n}{w_n^2 - w^2 + i\frac{2\xi}{\omega} w_n}. \rightarrow (13) \]

\[ \Rightarrow \text{called the impedance function.} \]

\[ \frac{Z(iw)e^{iwt}}{Z(iw)} \Rightarrow x(iw) = \frac{w_n^2 A}{Z(iw)} \]

\[ = \frac{w_n^2 A}{w_n^2 - \frac{w^2}{\omega_n^2} + i\frac{2\xi}{\omega_n} w} = \frac{A}{1 - \frac{w^2}{\omega_n^2} + i\frac{2\xi}{\omega_n} w} \rightarrow (14) \]

Non-dimensionalizing (14) gives:

\[ G(iw) = \frac{x(iw)}{A} = \frac{w_n^2}{Z(iw)} = \frac{1}{1 - \frac{w^2}{\omega_n^2} + i\frac{2\xi}{\omega_n} w} \]

\[ \Rightarrow \text{frequency response is a non-dimensional ratio.} \]

\[ \Rightarrow \text{very important in vibration.} \]

Thus,

\[ x(t) = A \cdot G(iw) e^{iwt} \rightarrow \text{general form of response.} \]

\[ G(iw) \rightarrow \text{system response to a harmonic excitation frequency } \omega. \]

This frequency response \( G(iw) \) is a complex function.

So,

\[ G(iw) = \text{Re} G(iw) + i \text{Im} G(iw) \]

for any complex number \( z \):

\[ z = a + ib \]

\[ |z| = |z| = |z| \quad z = |z| e^{-i\phi(z)} \]

\[ |G(iw)| = \left| G(iw) \cdot \overline{G(iw)} \right|^\frac{1}{2} \]

Complex conjugate of \( G(iw) \)

\[ \downarrow \]

Magnitude of \( G(iw) \)

\[ \text{of } G(iw) \]
\[ \phi(\omega) = \tan^{-1} \left[ \frac{-\text{Im} \, G(i\omega)}{\text{Re} \, G(i\omega)} \right] \]

Phase angle of \( G(i\omega) \).

Hence, \( x(t) = A \, G(i\omega) e^{i\omega t} \) becomes,
\[ x(t) = A \left| G(i\omega) \right| e^{i(\omega t - \phi)} \]

1. If \( f(t) = A \cos \omega t \),
\[ x(t) = \text{Re} \left[ A \left| G(i\omega) \right| e^{i(\omega t - \phi)} \right] \]
\[ = A \left| G(i\omega) \right| \cos(\omega t - \phi) \]

2. If \( f(t) = A \sin \omega t \),
\[ x(t) = \text{Im} \left[ A \left| G(i\omega) \right| e^{i(\omega t - \phi)} \right] \]
\[ = A \left| G(i\omega) \right| \sin(\omega t - \phi) \]

Geometric representation in Complex plane:
\[ i \text{Im} x(t) \]

\[ \begin{align*}
\dot{x}(t) &= i\omega A \left| G(i\omega) \right| e^{i(\omega t - \phi)} \\
\ddot{x}(t) &= (i\omega)^2 A \left| G(i\omega) \right| e^{i(\omega t - \phi)} \\
&= -\omega^2 x(t) \\
i &= \cos \frac{\pi}{2} + i \sin \frac{\pi}{2} = i \\
\text{Velocity leads the displacement } x(t) \text{ by } \frac{\pi}{2}. \\
\text{and Magnitude velocity } &= \omega x \text{ displacement } x(t) \\
-1 &= \cos \phi + i \sin \phi = e^{i\phi} \\
a \text{ acceleration leads the displacement by } \frac{\pi}{2} \text{ and magnitude of } a &= \omega^2 x(t) \]
Frequency Response plots.

Substitute, \( G(iw) = \frac{X(iw)}{Z(iw)} = \frac{\frac{w_n^2}{2} \left( 1 - \frac{w^2}{w_n^2} \right)}{Z(iw) - \frac{1}{2} \frac{w_n^2}{w_i} \left( 1 - \frac{w^2}{w_n^2} \right)} \).

\[ |G(iw)| = \left[ \frac{G(iw) \overline{G(iw)}}{|G(iw)|^2} \right]^{1/2} = \left[ \left( \frac{\Re G(iw)}{2} \right)^2 + \left( \frac{\Im G(iw)}{2} \right)^2 \right]^{1/2} \]

We get, \( |G(iw)|^2 = \left[ 1 - \left( \frac{w}{w_n} \right)^2 \right] \frac{1}{\left[ 1 - \left( \frac{w_n}{w} \right)^2 + \frac{2 \xi w}{w_n} \right]^{1/2}} \).

\[ G(iw) = G(iw) \frac{Z(iw)}{G(iw)} = \frac{|G(iw)|}{G(iw)} = \frac{1 - \left( \frac{w}{w_n} \right)^2}{\left[ 1 - \left( \frac{w}{w_n} \right)^2 + \frac{2 \xi w}{w_n} \right]^{1/2}} \]

\[ \Re G(iw) = \frac{1 - \left( \frac{w}{w_n} \right)^2}{\left[ 1 - \left( \frac{w}{w_n} \right)^2 + \frac{2 \xi w}{w_n} \right]^{1/2}} \]

\[ \Im G(iw) = -\frac{2 \xi}{\left[ 1 - \left( \frac{w}{w_n} \right)^2 + \frac{2 \xi w}{w_n} \right]^{1/2}} \]

\[ \phi(w) = \tan^{-1} \left( \frac{-\Im G(iw)}{\Re G(iw)} \right) = \tan^{-1} \left( \frac{2 \xi w}{1 - (w/w_n)^2} \right) \]

Non-dimensional plots \( |G(iw)| \) vs \( \frac{w}{w_n} \) and \( \phi(w) \) vs \( \frac{w}{w_n} \) are known as frequency response plots. Here damping factor \( \xi \) is used as a parameter.
damping tend to reduce the amplitudes and shift the peaks to left of the vertical through \( w/w_n = 1 \).

To get a peak value, 
\[
\frac{d}{d \left( w/w_n \right)} |Q(\omega)| = 0.
\]

\[
\left( \frac{1 - (w/w_n)^2}{2} \right)^2 + 2 \left( \frac{w}{w_n} \right)^2 \frac{\partial Q}{\partial w} = 0.
\]

\[
\Rightarrow \quad \frac{w}{w_n} = \sqrt{1 - 2\xi^2} \quad \text{once the peaks.}
\]

peaks occur when \( \frac{w}{w_n} < 1 \)

\( \xi > \frac{1}{\sqrt{2}} \), the response has no peaks.

For damping factor \( \xi = 0 \), (Undamped), the response increases indefinitely as \( w \to w_n \), called Resonance condition. (Violent vibration).

At \( \xi = 0 \), the solution of the eqn is no more valid as \( w \neq w_n \).
The value of the peak amplitudes,

\[ |g(i\omega)|_{\text{max}} = \frac{1}{2\sqrt{n - 3}} \]

light damping, \( \xi < 0.05 \), peaks occur very near to \( \omega_{\text{sh}} = 1 \) line

So for small values of \( \xi \), \( |g(i\omega)|_{\text{max}} = \frac{1}{2\xi} = Q \).

Where \( Q \) called quality factor.

(ex: in electrical engs, in tuning a circuit to the frequency to get nearly resonance amplitude.) \( \to \) factor of the circuit.

So to get viscous damping factor of a system \( \xi = \frac{1}{2Q} \).

where \( Q = \) peak amplitude.

\( P_1 \) \& \( P_2 \) are two points where amplitude of \( |g(i\omega)| \) reaches \( Q \sqrt{2} \)

called half power points

being power absorbed \( \propto (\text{amplitude})^2 \)

(ex: in electrical circuit)

To get \( P_1 \) \& \( P_2 \),

\[ |g(i\omega)| = \frac{Q}{\sqrt{2}} \]

\[ \frac{1}{\sqrt{1 - \left( \frac{\omega}{\omega_{\text{sh}}} \right)^2 + (2\xi \frac{\omega}{\omega_{\text{sh}}})}} = \frac{Q}{\sqrt{2}} \]

\[ (\frac{\omega}{\omega_{\text{sh}}})^4 - 2(1 - 2\xi^2)(\frac{\omega}{\omega_{\text{sh}}}) + 1 - 8\xi^2 = 0 \]

So \((\omega_{\text{sh}})^2 = y \), then \( y^2 - 2(1 - 2\xi^2)y + (1 - 8\xi^2) = 0 \)

a quadratic in \( y \) in \( (\omega_{\text{sh}})^2 \).

Ignore terms of \( \xi^3 \), the soln gives,

\[ \left( \frac{\omega_{\text{sh}}}{\omega_{\text{sh}}} \right)^2 = (1 - 2\xi^2) \pm \sqrt{(1 - 2\xi^2)^2 - (1 - 8\xi^2)} \]

\[ \left( \frac{\omega_{\text{sh}}}{\omega_{\text{sh}}} \right)^2 = 1 - 2\xi^2 \pm 2\xi \]
\[ a, \left( \frac{w_2}{\omega_n} \right)^2 - \left( \frac{w_1}{\omega_n} \right)^2 = 4 \xi^2. \]

For light damping, \( \xi \leq 0.05 \).

\[ w_1 + w_2 = 2 \omega_n \]

\[ \omega_n \]

\[ \Delta \omega = w_2 - w_1 = 2 \xi \omega_n \]

\[ \Delta \omega \]

Increment of frequency at half power points.

\[ G = \frac{1}{\Delta \omega} = \frac{\omega_n}{w_2 - w_1} \]

(Quality factor at small bandwidths).

\[ G \]

Plot \( q(w) \) vs. \( \frac{w}{\omega_n} \) with \( \xi \) as a parameter.

\[ q(w) \]

The curves pass through the point \( q = \pi/2, \frac{w}{\omega_n} = 1 \)

For \( \frac{w}{\omega_n} < 1 \), \( q \to 0 \) as \( \xi \to 0 \)

For \( \frac{w}{\omega_n} > 1 \), \( q \to \pi \) as \( \xi \to 0 \)

For \( \frac{w}{\omega_n} < 1 \), \( q = 0 \) for \( \xi = 0 \) and it experiences a discontinuity at \( \frac{w}{\omega_n} = 1 \) and then jumps from 0 to \( \pi/2 \) and then to \( \pi \)

And it continues with \( q = \pi \) for \( \xi = 0 \) for \( \frac{w}{\omega_n} > 1 \)
Degree of freedom \( \eta \) is the number of independent variables required to define the motion of the system.

When damping occurs, the natural frequency is insignificant and the corresponding frequency called damped frequency.

A rigid body have 3 DOF linear and 3 DOF w/ rotation.

A flexible beam can have theoretically infinite DOF as it is made of infinite number of particles have 3 DOF. But practically, we take some natural frequency of the beam at various operating modes.

French Mathematician Fourier said that, a periodic wave or function can be represented as, \( \sin \) or \( \cos \) waves

\[ x(t) = \frac{a_0}{2} + a_1 \cos \omega_1 t + a_2 \cos \omega_2 t + \ldots + b_0 \frac{\sin}{2} + b_1 \sin \omega_1 t + b_2 \sin \omega_2 t + \ldots \]

\[ \omega = \frac{2\pi}{T} \]

\[ \omega_n = \eta \omega_1 \]

Degree of Freedom 

\[ \eta = 3N - C \]

\( N \) : No. of particles

\( C \) : No. of constraints

SHM
Bending Vibration or Transverse Vibration.

\[ W_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{BEI}{mL^3}} \]

Axial Vibration

\[ S = \frac{P}{A} \quad , \quad k = \frac{P}{S} \]

\[ W_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{AE}{lm}} \]

Torsional Vibration

\[ \frac{T}{J} = \frac{G\theta}{l} \quad \text{angular twist} \]

\[ K_t = \frac{T}{\theta} = \frac{GJ}{l} \]

\[ W_n = \sqrt{\frac{K_t}{J}} = \sqrt{\frac{GJ}{J_l^2}} \]

\[ k = \frac{Gd^4}{64nR} \]

\[ k = \frac{EI}{l} \]

\[ \theta (\theta) \]

\[ I = \text{moment of inertia of cross section curve} \]

\[ L = \text{total length of the strip} \]

\[ W_n = \sqrt{\frac{k}{J_p}} \]

\[ k = \frac{EI}{L} \]

\[ S = \frac{P}{BL^3} \quad \text{length} \]

\[ \text{Young's Modulus} \]

\[ \text{from strength of materials} \]

\[ K = \frac{\text{Load}}{\text{deformation}} = \frac{P}{\frac{PL^3}{3EI}} = \frac{3EI}{R^3} \]
Now, $\theta \to 0$

Phase angle at resonance is $\pi/2$.

At $\theta = 0$, the diff. eqn will not have odd-order derivatives.

No complex terms.

$x(t) + \omega_n^2 x(t) = \omega_n^2 A \cos \omega t$.

Let the sln, $C_1 = 0$.

the response, $x(t) = \frac{1}{1 - (\omega/\omega_n)^2} A \cos \omega t$. \(\omega_n \text{ Real fn.} \)

$\Rightarrow C_1(\omega) = \frac{1}{1 - (\omega/\omega_n)^2}$.

$G(\omega) = \text{Im} \text{ for } \frac{\omega}{\omega_n} < 1$

response is in phase with excitation.

$G(\omega) = -\omega_n \text{ for } \omega_n > 1$

Response is 180° out of phase with the excitation, for $\omega > \omega_n$. Curvature.

So at $\omega = \omega_n$ (i.e. Resonance),

$x(t) = \frac{A}{\omega_n} A G(\omega) \cos \omega t$ is no more valued.

as it gives infinite response. So we assume that the sln be

$x(t) = \frac{A}{2} \omega_n t \sin \omega nt$. (verified by substitution). Response is oscillating

with an amplitude increasing linearly with time.
Straight lines $\pm \frac{A \omega t}{2} \rightarrow$ linearly widening envelope. (Wid fluctuations)

So here we assume the excitation is proportional to $\cos \omega t u$ while the response is $\alpha$ to $\sin \omega t u$. But $\sin \omega t u = \cos (\omega t u - \pi/2)$.

$\alpha$, response and stimulation differ by $\pi/2$ phase angle.
Systems with Rotating Unbalanced Masses:

Ex: A rotating unbalanced mass is Waston's machine
with nonuniformly distributed clothes.

\[ \sum \vec{F} = \frac{\text{Net external force}}{m} \]

\[ \vec{a}(t) + \vec{c}(t) + \vec{k} \cdot \vec{x}(t) = \frac{\text{Net force}}{m} \]

\[ \vec{a}(t) = \vec{c}(t) + \vec{k} \cdot \vec{x}(t) \]

Total mass = Mass of Dynamometer + Masses of Displacement

\[ \frac{\text{Constant}}{m} = \frac{\text{Constant}}{\text{Mass}} \]

\[ c = 2 \pi f_0 n, \quad \frac{k}{m} = n^2 \]

\[ \ddot{x}(t) + 2 \pi f_0 n \dot{x}(t) + \omega_0^2 x(t) = \frac{m}{\text{Mass}} \omega^2 \sin \omega t \]

Hooke's law is proportional to Sin \omega t

Let \(|G(i\omega)|\) be the magnitude and \(\phi(i\omega)\) is the phase angle.

Comparing the Eqs with \(\text{Analogous Approach}\)

\[ m \ddot{x} = Re \frac{A}{G(i\omega)} e^{i(\omega t - \phi)} = \frac{A}{G(i\omega)} \cos(\omega t - \phi) \]

of excitation is \(x(t) = A \sin \omega t\),

\[ x(t) = \text{Im} \frac{A}{G(i\omega)} e^{i(\omega t + \phi)} = \frac{A}{G(i\omega)} \sin(\omega t + \phi) \]

\[ \frac{m}{\text{Mass}} \omega^2 = \omega_0^2 n \]

Hence \(A = \left(\frac{m}{\text{Mass}}\right) e^{\frac{\omega^2}{\omega_0^2 n}}\)

\[ \ddot{x}(t) = \frac{m}{\text{Mass}} e^{\left(\frac{\omega^2}{\omega_0^2 n}\right)} \cdot \frac{|G(i\omega)|}{\text{Sin}(\omega t - \phi)} \]

\[ x(t) = \frac{1}{\text{Sin}(\omega t - \phi)} \]

\[ \frac{M}{\text{Mass}} \cdot \frac{1}{\text{Sin}(\omega t - \phi)} = \frac{\text{Sin}(\omega t - \phi)}{\text{Sin}(\omega t - \phi)} \]

\[ \frac{G(i\omega)}{Z(i\omega)} = \frac{A}{Z(i\omega)} \]

\[ Z(i\omega) = \frac{\omega_0^2 n}{\text{Mass}} \]

\[ G(i\omega) = \frac{\omega^2}{\omega_0^2 n} \]

\[ \frac{M}{\text{Mass}} = \left(\frac{\omega}{\omega_0 n}\right)^2 |G(i\omega)| \]

\[ Z(i\omega) = \frac{\omega_0^2 n}{\text{Mass}} \]
The frequency response plots.

Here:

1) All curves begin at 0 instead of 1.
2) All peaks occur for $\frac{\omega}{\omega_n} > 1$ instead of $\frac{\omega}{\omega_n} = 1$.
3) As $\frac{\omega}{\omega_n}$ becomes very large, the magnitude of the response tends to $1$. Instead of tending to $0$.

Why the magnitude of response tend to $1$.

The position of system mass centre for large $\frac{\omega}{\omega_n}$ is determined.

$a$, The main mass $(m - m')$ undergoes a displacement $x$ and the eccentric mass $m'$ undergoes the vertical displacement $x + \varepsilon \sin \omega t$.

System with rotating eccentric Mass.

FBD for Main Mass. $m'(\dot{\varepsilon} + \sin \omega t)$

FBD for Eccentric Mass. $m'\varepsilon \sin \omega t$
\[
\lim_{x \to 0} \frac{1}{x} \left( \frac{\sin mx + \sin nx}{x} \right) = \frac{1}{x} \left( \frac{\sin mx - \sin nx}{x} \right) = \frac{\sin mx}{x} - \frac{\sin nx}{x} = m - n
\]

\[
\lim_{x \to 0} \frac{1}{x^2} \left( \frac{\sin mx + \sin nx}{x} \right) = \frac{1}{x^2} \left( \frac{\sin mx - \sin nx}{x} \right) = \frac{\sin mx}{x^2} - \frac{\sin nx}{x^2} = \frac{m}{x} - \frac{n}{x} = m - n
\]

\[
\lim_{x \to 0} \frac{1}{x^3} \left( \frac{\sin mx + \sin nx}{x} \right) = \frac{1}{x^3} \left( \frac{\sin mx - \sin nx}{x} \right) = \frac{\sin mx}{x^3} - \frac{\sin nx}{x^3} = \frac{m}{x^2} - \frac{n}{x^2} = m - n
\]

\[
\lim_{x \to 0} \frac{1}{x^4} \left( \frac{\sin mx + \sin nx}{x} \right) = \frac{1}{x^4} \left( \frac{\sin mx - \sin nx}{x} \right) = \frac{\sin mx}{x^4} - \frac{\sin nx}{x^4} = \frac{m}{x^3} - \frac{n}{x^3} = m - n
\]
Whirling of Rotating Shafts

Rotors \rightarrow \text{Rotating part/disk of a Mechanical System.}

Connected to flexible shafts \rightarrow \text{mounted on bearings}

x: Turbines, Compressors, etc.

If rotors have some eccentricity, the mass centre of the disc donot coincide with geometric centre, then the rotation produces a centrifugal force causing the shaft to bend. The rotation of the plane containing the bent shaft about the bearing axis is known as whirling.

For certain rotational velocities \rightarrow \text{violent vibrations to the system.}

w \rightarrow \text{const angular velocity}

m \rightarrow \text{disc of mass ''m''}

No mass for shaft.

x, y is the plane rotation occurs

S \rightarrow \text{geometric centre of disc}

2 DOF System (independent) \rightarrow 8 \text{ DOF System with 1 DOF each.}

O \rightarrow \text{origin of x-y}

C \rightarrow \text{mass centre of disc.}

C' does not coincide with S

displacement between S & C is 'e' \rightarrow \text{eccentricity}

\( a_c \rightarrow \text{acceleration of mass centre (}
\)

\( \mathbf{a}_c = \text{acceleration vector of } \mathbf{C} \)

\[ \mathbf{a}_c = (x + e \cos wt) \mathbf{i} + (y + e \sin wt) \mathbf{j} \]

\[ \frac{d^2 \mathbf{r}}{dt^2} = \mathbf{a}_c = \left( x + e \omega^2 \cos wt \right) \mathbf{i} + \left( y + e \omega^2 \sin wt \right) \mathbf{j} \]

\[ \mathbf{a}_c = \frac{d^2 \mathbf{r}}{dt^2} = \left( x + e \omega^2 \cos wt \right) \mathbf{i} + \left( y - e \omega^2 \sin wt \right) \mathbf{j} \]

\text{Forces: (1) Restoring forces due to elasticity of shaft.}

\text{(2) Resisting forces due to viscous damping. (Due to air friction)}

\text{Elastic forces represented by Spring constants } k_x \text{ & } k_y \text{ associated with deformation of shaft.}

m \cdot x, y \text{ dynamic}
Coefficient of viscous damping, $c$, and some id. $x$, $y$ directions.

Elastic restoring forces, damping forces are acting at $s$.

By Newton's Law,

$$-k_x x - C_x = m \left( \ddot{x} - e^{\alpha t}\cos\omega t \right)$$

$$-k_y y - C_y = m \left( \ddot{y} - e^{\alpha t}\sin\omega t \right)$$

$$m\ddot{x} + \omega^2 \cos\omega t + C_x + k_x x = 0$$

$$m\ddot{y} + \omega^2 \sin\omega t + C_y + k_y y = 0$$

$$m\ddot{x} + C_x + k_x x = m e^{\omega t}\cos\omega t$$

$$m\ddot{y} + C_y + k_y y = m e^{\omega t}\sin\omega t$$

By $m = \frac{C}{\omega^2}$, $x(0) = 0$, $\dot{x}(0) = 0$, $y(0) = 0$, $\dot{y}(0) = 0$

$$x(t) = x(\omega) \cos(\omega t - \phi_x)$$

$$y(t) = y(\omega) \sin(\omega t - \phi_y)$$

$$|x(\omega)| = e^{-\left(\frac{\omega^2}{\omega^2_{nx}}\right)} |G_x(\omega)|$$

$$|y(\omega)| = e^{-\left(\frac{\omega^2}{\omega^2_{ny}}\right)} |G_y(\omega)|$$

where,

$$|G_x(\omega)| = \left\{ \left[ 1 - \left(\frac{\omega}{\omega_{nx}}\right)^2 \right]^2 + \left( 2 \frac{\omega}{\omega_{nx}} \right)^2 \right\}^{\frac{1}{2}}$$

$$|G_y(\omega)| = \left\{ \left[ 1 - \left(\frac{\omega}{\omega_{ny}}\right)^2 \right]^2 + \left[ 2 \frac{\omega}{\omega_{ny}} \right]^2 \right\}^{\frac{1}{2}}$$

$$\phi_x = \tan^{-1}\left( \frac{2 \omega}{\omega_{nx}} \right)$$

$$\phi_y = \tan^{-1}\left( \frac{2 \omega}{\omega_{ny}} \right)$$

For a shaft of circular cross section, $k_x = k_y = k$

$$\omega_{nx} = \omega_{ny} = \omega = \sqrt{\frac{k}{m}}$$

$$\omega_{nx} = \omega_{ny} = k = \frac{C}{2m}$$

Hence,

$$|G_x(\omega)| = |G_y(\omega)| = \left| \frac{1}{\omega^2} \right|^2 \left\{ \left[ 1 - \left(\frac{\omega}{\omega_{nx}}\right)^2 \right]^2 + \left( 2 \frac{\omega}{\omega_{nx}} \right)^2 \right\}^{\frac{1}{2}}$$
So Amplitude of motion are,

\[ |x(w)| = |y(w)| = e^{-\left(\frac{\omega}{\omega_n}\right)^2} |q(i\omega)| \]

\[ \tan \theta = \frac{y}{x} = \tan (\omega t + \phi) \]

\[ \theta = \omega t - \phi \]

\[ \phi = \tan^{-1} \left(\frac{y}{x}\right) \quad \text{(called Synchronus whirl)} \]

Shalf + disc \implies \text{Rigid body}. \quad \text{(Called Synchronus whirl)}

The radial distance from 0 to S

\[ R_{os} \]

for given \( \omega \) is a constant for

\[ \text{Synchronus whirl} \]

\[ R_{os} = \sqrt{x^2 + y^2} \]

\[ e^{-\left(\frac{\omega}{\omega_n}\right)^2} \quad |q(i\omega)| = \text{constant} \]

So that \( S \) makes a circle, as centre 0.

\[ \phi = \text{angle between } R_{os} \text{ and } R_{st} \]

position of \( C \) is relative to whirlings plane.

So from \( \phi = \tan^{-1} \left(\frac{y}{x}\right) \), we get \( \phi < \frac{\pi}{2} \) for \( \omega < \omega_n \)

\[ \phi = \frac{\pi}{2} \text{ for } \omega = \omega_n \]

\[ \phi > \frac{\pi}{2} \text{ for } \omega > \omega_n \]

\[ \text{Case 1} \] \quad \[ k_x = k_y \]

\[ x(t) = x(w) \cos \omega t \]

\[ y(t) = y(w) \sin \omega t \]

\[ x(w) = e^{-\left(\frac{\omega}{\omega_n}\right)^2} \quad \frac{\omega}{\omega_n} \]

\[ y(w) = \frac{\left(\frac{\omega}{\omega_n}\right)^2}{1 - \left(\frac{\omega}{\omega_n}\right)^2} \]

\[ \text{Using undamped} \]

\[ x(t) = x(w) \cos \omega t \]
Divide \( x(t) \) and \( y(t) \), then Square and add.

\[
\frac{x^2}{x^2} + \frac{y^2}{y^2} = 1
\]

(representing equation of an ellipse)

\( u \), when the shaft is at \( x \), point \( S \) traverses an ellipse with point \( O \) as its geometric centre.

\[
\tan \theta = \frac{y}{x} = \frac{y}{x}
\]

We know \( \tan \theta = \frac{y}{x} \tan \omega t \).

Differentiate w.r.t. time \( t \), then substitute \( x(t) = x(w) \cos \omega t \) and \( y(t) = y(w) \sin \omega t \).

\[
\dot{\theta} = \frac{xy}{(x^2 \cos^2 \omega t + y \sin^2 \omega t)}
\]

So the sign of \( \dot{\theta} \) depends on \( x > 0 \) and \( y > 0 \), \( xy > 0 \). So the sign of \( \dot{\theta} \) depends on \( x > 0 \) and \( y > 0 \), \( xy > 0 \).

Case 1. \( w > w_{nx} \) and \( w < w_{ny} \), \( xy > 0 \), \( S \) moves on ellipse in the same direction as \( w \).

Case 2. \( w > w_{nx} \) and \( w > w_{ny} \)

Case 3. \( w > w_{nx} \) and \( w > w_{ny} \), \( xy > 0 \), \( S \) moves in the same sense of \( w \).
Since $C = 0$, possibility of resonance at $W = W_{nx}$.

At resonance, solution is not valid.

Method of Substitution:

$x(t) = \frac{1}{2} e^{W_{nx} t} \sin W_{nx} t$

$y(t) = -\frac{1}{2} e^{W_{nx} t} (\cos W_{nx} t - \frac{1}{2} W_{nx} e^{W_{nx} t} \sin W_{nx} t)$

Plot of $x(t)$ vs $t$.

Plot of $y(t)$ vs $t$.

The frequencies $W_{nx} \neq W_{ny}$ are called critical frequencies or critical speed.

\[ \sin (W_{ny} t - \frac{1}{2}) = -\cos W_{ny} t \]

\[ W_{nx} < W_{ny}, \quad W_{nx} < W_{ny}, \quad W_{ny} < W_{ny} < W_{nx}, \quad W_{ny} > W_{ny} \quad W_{ny} > W_{nx} \]

$W \rightarrow$ shaft angular velocity

$W \rightarrow$ rotor angular velocity

$\theta$
Harmonic Motion of the Base:

To protect the equipment, it is necessary to isolate the equipment from the damaging effects of vibrating foundations. The use of rubber mounts act as springs & dampers in the same manner as the support of W, M.

Vehicle on a Wavy Road $\Rightarrow$ Suspension must be isolated from other parts of the body.

Engine Mounted on a Vibrating Aircraft Wing.

\[ x(t) \]

\[ y(t) \]

Mass - damper - spring system moving with base.

\[-c(x-y) - k(x-y) = m\ddot{x}\]

Dividing by $m$ and rearranging,

\[ \ddot{x} + 2\xi\omega_n\dot{x} + \omega_n^2 x = 2\xi\omega_n y + \omega_n^2 y \Rightarrow 0 \]

Understand that motion of base is harmonic,

\[ y(t) = \text{Re } A e^{j\omega t} \rightarrow \text{displacement of base} \]

and

\[ x(t) = X(i\omega) e^{i\omega t} \rightarrow \text{Response of the system} \]

If excitation is $A\cos\omega t$, Response is $\text{Re } x(t)$

A Siant: $^{\text{Im } x(t)}$

1. Dividing through by $e^{i\omega t}$, solve for $x(i\omega)$ we get,

\[ X(i\omega) = \frac{1 + i2\xi\omega_n}{1 - (\frac{\omega_n}{\omega})^2 + i2\xi}\frac{\omega_n}{\omega} \]

In which $g(i\omega)$ is the frequency response.

We know,

\[ X(i\omega) = |X(i\omega)| e^{-i\phi(\omega)} \]

\[ x(t) = |X(i\omega)| e^{i(\omega t - \phi)} \]
The Analytical Approach:

\[ |X(i\omega)| = \sqrt{\frac{X(i\omega)}{\sqrt{\frac{1}{(1 + i2\pi \omega \frac{\omega}{\omega_n})^2 - 1}}}} \cdot A \]

\[ = \left[ 1 + \left( \frac{\omega}{\omega_n} \right)^2 \right]^{1/2} \cdot \frac{|G(i\omega)|}{A} \]
In which $G(\omega)$

\[ X(\omega) = X(i\omega) \frac{\overline{G}(i\omega)}{G(i\omega)} \]

\[ = (1 + i2\frac{\omega}{\omega_n} + i2\frac{\omega}{\omega_n}) \left[ 1 - \left( \frac{\omega}{\omega_n} \right)^2 - i2\frac{\omega}{\omega_n} \right] G(i\omega) A \]

\[ = \left[ 1 - \left( \frac{\omega}{\omega_n} \right)^2 + i2\frac{\omega}{\omega_n} \right] G(i\omega) A \]

by analogy,

\[ \phi(\omega) = \tan^{-1} \left( \frac{-\Im X(i\omega)}{\Re X(i\omega)} \right) = \tan^{-1} \left( \frac{\omega^2 \left( \frac{\omega}{\omega_n} \right)^2}{1 - (\omega^2) + (\omega^2 \frac{\omega}{\omega_n})^2} \right) \]

\[ X(\omega) = \left| X(i\omega) \right| e^{i(\omega t - \phi)} \]

represents a steady-state harmonic response.

The nondimensionalized ratio of response, \( A \)

\[ \left| \frac{X(i\omega)}{A} \right| = \left[ 1 + \left( \frac{2\frac{\omega}{\omega_n}^2}{1 + (2\frac{\omega}{\omega_n})^2} \right)^{1/2} \right] \left| \frac{X(i\omega)}{A} \right| = \text{known as the transmissibility} \]

\[ \left| G(i\omega) \right| = \left\{ \left[ 1 - \left( \frac{\omega}{\omega_n} \right)^2 \right] + \left[ 2\frac{\omega}{\omega_n} \right]^2 \right\}^{1/2} \]

Substitute $\frac{\omega}{\omega_n} = \sqrt{2}$

\[ \Rightarrow \left| G(i\omega) \right| = \left[ 1 + \left( \frac{2\sqrt{2}}{2} \right)^2 \right]^{1/2} \]

\[ \frac{1 + \left( \frac{2\sqrt{2}}{2} \right)^2}{1 + \left( \frac{2\sqrt{2}}{2} \right)^2} = 1^{1/2} = 1 \]

\( \omega_n \) Response has the same magnitude as the excitation. It is magnified relative to the excitation for $\frac{\omega}{\omega_n} < \sqrt{2}$ and reduced for $\frac{\omega}{\omega_n} > \sqrt{2}$.

The amount of magnification or reduction depends on \( S \)
Vibration Isolation

In many systems, we are interested in transmitting as little vibration as possible to the base. But this will become critical when the excitation is harmonic.

Force is transmitted to the base through springs & dampers, so we conclude that the amplitude of the force is:

$$F_{tx} = m \left[ (2\pi \frac{w_n}{w})^2 + (\frac{w_n}{w})^2 \right]^{1/2}$$

Amplitude of velocity $\dot{v} = \frac{w_n}{2\pi}$

$$m \frac{w_n^2}{w_n} = k$$

$$F_{tx} = k \left[ 1 + \left( \frac{2\pi \frac{w_n}{w}}{w_n} \right)^2 \right]^{1/2}$$

phase angle of force is same,

$$F_{tx} = \frac{A}{k} \left[ 1 + \left( \frac{2\pi \frac{w_n}{w}}{w_n} \right)^2 \right]^{1/2} \text{ for } \theta(\omega)$$

Amplitude of the actual excitation force $= F_0$

Non dimensionalizing:

$$\frac{F_{tx}}{F_0} = \left[ 1 + \left( \frac{2\pi \frac{w_n}{w}}{w_n} \right)^2 \right]^{1/2} \text{ for } \theta(\omega)$$

$$\frac{F_{tx}}{F_0} = \text{ Non dimensional ratio of the force transmitted to the base,}$$

$$\text{called transmissibility}$$

So plots $\frac{F_{tx}}{F_0}$ vs $\frac{w}{w_n}$ is same as plots of $|X|$ vs $\frac{w}{w_n}$

When $\frac{w}{w_n} = \sqrt{2}$, full force is transmitted to base i.e., $\frac{F_{tx}}{F_0} = 1$

For $\frac{w}{w_n} > \sqrt{2}$, the force transmitted tends to decrease with increasing driving frequency $\omega$, regardless of $\xi$.

In fact, $\xi$ (dampers) do not alleviate the situation.

For $\frac{w}{w_n} > \sqrt{2}$, transmitted force increases as dampers increase.
Isolation Efficiency = 1 - T

\[
\frac{w}{\omega_n} \rightarrow T = 0
\]

Region of Isolation:
T < 1
Isolation occurs at frequency ratio 1.4

Natural frequency of isolator
- Frequency factor 3 (10 dB) attenuation

\[
T = 1, \text{No reduction at Amplitude.}
\]

\[
T = 0.1, \text{Next highest frequency.}
\]
90% reduction at Amplitude of input vibrations.

\[
T = 0.01, \text{Next highest frequency.}
\]
99% reduction at Amplitude of input vibrations.

99.9% efficient, 100 dB attenuation

At frequency Hz.
\[ T = \frac{\text{Amplitude of vib. response}}{\text{Amplitude of vib. input} \sqrt{1 + \left(2\delta \frac{w}{w_n}\right)^2}} \] 

\[ \delta = \text{damping ratio factor} \]
\[ w \rightarrow \text{Driving force} \]
\[ w_n = \text{Natural force} \]

\[ T = \% \text{ of vib. energy that is being transmitted through a system} \]

Material trans. curve.
**Isolation Theory**

The use of Isolation is primarily for reducing the effect of the dynamic forces generated by moving parts in a machine into the surrounding structure.

- **Vibration Control**
  - Weight to be supported
  - Damping frequency of the machine
  - Rigidity of the structure supporting the machine

The transmission of v.i.b. force is automated effectively by using Resilient Material; when subjected static/dynamic loads, establishes Not-freq. of the Isolation System.

leads to extensive study on material to absorb vibration.

**Vib. measuring Instruments**

- Measure displacements & accelerations.
- Mass - Damper - Spring System.
- Transducer - measure displacements.

**Energy Converter**

Mass \( \rightarrow \) called Seismic Mass or proof Mass - constrained to move.

along or given axes.

Damping - by a viscous fluid.

Displacement of Mass relative to case and the absolute displacement of the Mass on \( x(t) \), \( y(t) \) and \( z(t) \).

\[
x(t) = y(t) + z(t)
\]

\[
m \ddot{x}(t) + c \left[ \dot{x}(t) - \dot{y}(t) \right] + k \left[ x(t) - y(t) \right] = 0
\]

eliminate \( x(t) \) \( \rightarrow \) absolute disp.

\[
m \ddot{z}(t) + c \dot{z}(t) + k z(t) = - m \ddot{y}(t)
\]
Isolation Theory

The use of isolation is primarily for reducing the effect of the dynamic forces generated by moving parts in a machine into the surrounding structure.

Vibration Control

1. Weight to be supported
2. Distorting frequency of the machine
3. Rigidity of the structure supporting the machine

The transmission of vib. force is actuated opposed effectively by using Resilient Material. When subjected static/dynamic loads, it establishes Nut-freq. of the isolation system.

→ Leads to extensive study. on material / to absorb vibration.

Vib. measuring Instruments:

- Measure displacements & accelerations.
- Mass - damper - spring system.
- Transducer - measure displacements.
- Energy converter: (mechanical to electrical)

Mass → called Seismic Mass or proof mass. constrained to move along a given axis.
Damping → by a viscous fluid.

Displacement of Mass relative to case and the absolute displacement of the Mass on \( y(t) \), \( z(t) \) and \( x(t) \).

\[
x(t) = y(t) + z(t)
\]

\[
m \ddot{x}(t) + c [ \ddot{x}(t) - \dot{y}(t) ] + k [ x(t) - y(t) ] = 0
\]

Eliminate \( x(t) \) → absolute disp.

\[
m \ddot{z}(t) + c \ddot{z}(t) + k z(t) = - m \ddot{y}(t)
\]
Vibration measured as harmonics:

\[ y(t) = y_0 \cos \omega t \quad \text{(Assume)} \]

then:

\[ m^2 \ddot{y} + \ddot{y} + k_2 = y_0 \omega^2 \cos \omega t \]

by analogy:

\[ Z(t) = y_0 \left( \frac{\omega}{\omega_n} \right)^2 \left| G(\omega) \right| \cos (\omega t - \phi) \]

\[ \omega \]

\[ Z(t) = Z_0 \cos (\omega t - \phi) \]

\[ Z_0 \rightarrow \text{measurement Amplitude} \]

\[ \frac{Z_0}{y_0} = \left( \frac{\omega}{\omega_n} \right)^2 \left| G(\omega) \right| \]

Plot \( \frac{Z_0}{y_0} \) vs \( \frac{\omega}{\omega_n} \)

Accelorimeters (high frequency instruments)

For small values of the ratio \( \frac{\omega}{\omega_n} \), the magnitude \( |G(\omega)| \rightarrow 1 \)

So:

\[ Z_0 = y_0 \left( \frac{\omega}{\omega_n} \right)^2 \]

\[ y_0 \omega^2 \rightarrow \text{acceleration} \]

\[ Z_0 \rightarrow \text{amplitude} \]

\[ \frac{1}{\omega_n^2} \rightarrow \text{proportionality constant} \]

An approximated ratio:

\[ \frac{Z_0}{y_0} \approx \left( \frac{\omega}{\omega_n} \right)^2 \]

\[ \omega_n \gg \omega \]

\[ \frac{y_{\text{top}}}{y_{\text{bot}}} \rightarrow \text{Amplitude Ratio} \]

\[ \frac{y_{\text{top}}}{y_0} \approx \left( \frac{\omega}{\omega_n} \right)^2 \]

\[ (4) \]

\[ \frac{d^2y}{dt^2} + \left( \frac{1}{\omega_n^2} \right) y = 0 \]

\[ \frac{d^2y}{dt^2} + \left[ \left( \frac{1}{\omega_n^2} \right) - \left( \frac{1}{\omega_n^2} \right) \right] y = 0 \]

\[ y_{\text{top}} \sim \sqrt{\frac{1}{2}} y_{\text{bot}} \]