Response to Arbitrary Excitations

Convolution Integral:

All the above excitations have one thing in common: they are explicit for's of time.

Trapezoidal Rule \( \rightarrow \) linear combinations of step & ramp funs.

If Arbitrary excitation periodic \( \rightarrow \) Represent by a Fourier Series.

Then we insist a limiting process \( \rightarrow \) the period is allowed to approach infinity. \( \rightarrow \) so the essence of fun becomes periodic & arbitrary than periodic. \( \rightarrow \) so P-S becomes Fourier Integral. \( \rightarrow \) This is the frequency-domain representation of funs. \( \rightarrow \) More suitable for random excitations than for deterministic excitations.

2nd Approach: Arbitrary excitation as a Superposition of Impulses of varying magnitude and applied at different times.

This is a time-domain rep. of funs.

Consider arbitrary excitation \( F(t) \)

Response of an impulse corresponding to the time interval \( T < t < T + \Delta T \)

Time Increment \( \Delta T \) is sufficiently small that \( F(t) \) does not change very much over this time increment. The shaded area \( \Delta T \) the impulse acting over

\( T < t < T + \Delta T \) and having a magnitude \( F(t) \cdot \Delta T \).
Excitation Corresponds to the shaded area, as an impulsive force,

\[ F(t) \delta(t-T) = F(t) \Delta T \delta(t-T) \]

For a linear time invariant system,

Response, \( \Delta x(t, T) = F(t) \Delta T g(t-T) \)

so the approximated responses,

\[ x(t) = \sum_{n} F(t) \Delta T g(t-nT) \]

for exact response, \( x(t) = \int F(t) g(t-T) \, dt \)

The response is called convolution integral.

Lg Response as a superposition of impulse responses.

In convolution integral, the response is a superposition a fn of \( t-T \) rather than of \( T \), where \( T \) is the variable of integration.

To obtain \( g(t-T) \) from \( g(t) \)

- 2 operations are carried out:
  1) Shifting
  2) Folding

\[ F(t) \Delta T g(t-T) \]
The 2nd version of convolution integral in which the shifting and folding operations are carried out on \( F(t) \) instead of \( g(t) \).

To derive 2nd version of convolution integral:

we introduce a transformation of variable from \( t \) to \( \lambda \)

\[
\begin{align*}
\lambda &= t - \tau \\
\tau &= \lambda + t \\
d\tau &= d\lambda
\end{align*}
\]

\[
X(t) = \int_{0}^{\infty} F(\lambda) g(t - \lambda) d\lambda \quad \rightarrow (1)
\]

Substituting, 

\[
X(t) = \int_{0}^{t} F(t - \lambda) g(\lambda) d\lambda = \int_{0}^{t} F(t - \lambda) g(\lambda) d\lambda \rightarrow (2)
\]

\( t \) and \( \lambda \) are dummy variables of integration.

Combine (1) & (2),

\[
X(t) = \int_{0}^{t} F(\lambda) g(t - \lambda) d\lambda = \int_{0}^{t} F(t - \lambda) g(\lambda) d\lambda
\]

The convolution integral is symmetric in the excitation \( F(t) \) and the impulse response \( g(t) \) in the sense that the result is same regardless of which of the two functions is shifted and folded.

The choice of which form of convolution integral to be used depends on the nature of the function \( F(t) \) and \( g(t) \) and ease of integration.
Geometrical Interpretation:

- Arbitrary excitation
- Impulse response
  - Underdamped MDOF System
  - $t$ replaced by the variable of integral
  - Folding results in $g(t-T)$
    - Taking mirror image of
      - $g(t+T) = g(t+T)$ w.r.t. vertical axis, which amounts to replacing $T$ by $-T$.

Integration of the curve $F(t) g(t-T)$ gives area:

- Multiply $F(t)$ by $g(t-T)$
- Area = $x(t)$
Explain \( f(t) \) as an \( F(t) \) or \( F(t) + g(t) \).

Impulse response as \( g(t) \)
response as \( x(t) \).

Generally, the Convolution Integral involves two arbitrary functions \( f_1(t) \) and \( f_2(t) \). There are not necessarily be \( F(t) \) and \( G(t) \).

General form of the Convolution Integral demonstrated by means of the Laplace Transformation Method.

Laplace Transformation:

- Tool to study the response of linear systems with constant coefficients.

\[ L \cdot T \] used to transform a complicated problem into simpler ones.

1. Solve the simpler problems.
2. Then perform inverse L.T.

Used to solve initial value problems.

Systems behavior is defined by Ordinary D.E. to be satisfied for all positive times.

Definition of the L.T

We consider \( f(t) \) for \( f(t) \) defined for all values of time longer than zero, \( t > 0 \), and define the (One-Sided) L.T of \( f(t) \) by the Integral,

\[ L \{ f(t) \} = F(s) = \int_0^\infty e^{-st} f(t) dt \]

\( e^{-st} \) is the kernel of the transformation.

\( s \) is the Subsidiary Variable (generally a complex quantity).
Associated Complex plane. Called the S plane.

For Laplace Plane (at times)

Most ans describing physical phenomena satisfy these
Conditions.

Laplace transform is an Integral Transform perhaps second only to
the Fourier Transform in its utility in solving physical problems.

The Laplace transform is particularly useful in solving linear
Ordinary Diff. Eqns.

The (unilateral) (one-sided) L.T denoted by \( \mathcal{L} \) is defined by

\[
\mathcal{L} \left[ f(t) \right] (s) = \int_0^\infty f(t) e^{-st} \, dt
\]

where \( f(t) \) is defined for \( t \geq 0 \)

The unilateral L.T is almost generally called L.T

The bilateral L.T is defined by,

\[
\mathcal{L} \left[ f(t) \right] (s) = \int_{-\infty}^{\infty} f(t) e^{-st} \, dt
\]

The Inverse L.T called Bromwich Integral (B.I)

or Fourier - Mellin - Integral.

Due to Duhamel's convolution principle

L.T converts integral and differential eqns into algebraic eqns

In complex notation

\( \mathbb{C} \ni z = x + iy \) or \( x + iy \)

\( \Re(z) \rightarrow \text{Reals or (complex axis)} \)

\( \Im(z) \rightarrow \text{Imaginary or (complex axis)} \)

\( J = \sqrt{-1} \rightarrow \text{engineering notation} \)

\( J = \sqrt{-1} \rightarrow \text{general notation} \)
Interesting

- the time domain (u, Signals)
- the frequency domain (u, Fourier)

Differential in one domain corresponds to multiplication by e^{-st} in the other.

Multiplication by an exponential in one domain corresponds to a shift (or delay) in the other.

Transformation of Derivatives

\[
L \left \{ \frac{d^2 f(t)}{dt^2} \right \} = \int_0^\infty e^{-st} \frac{d^2 f(t)}{dt^2} \, dt = -s^2 F(s) \quad u, f(0) = 0
\]

Following, same pattern,

\[
L \left \{ \frac{d^3 f(t)}{dt^3} \right \} = \int_0^\infty e^{-st} \frac{d^3 f(t)}{dt^3} \, dt = -s^3 F(s) \quad u, f(0) = 0
\]

Transformation of Ordinary Diff. Eqn.

\[
M \frac{d^2 x(t)}{dt^2} + c \frac{dx(t)}{dt} + kx(t) = f(t)
\]

\[L \{x'(t)\} = X(s)
\]

\[x(t) \rightarrow \text{Initial displacement}
\]

\[x'(t) \rightarrow \text{Initial velocity}
\]

\[L \{x'(t)\} = F(s)
\]

\[M \left [ s^2 X(s) - sX(0) - \dot{x}(0) \right ] + c \left [ sX(s) - x(0) \right ] + kX(0) \Rightarrow X(s)
\]
Patterns.

Interesting symmetric patterns between:
- the time domain (u, Signals).
- the frequency domain (u, Their L-T)

Differentiation in one domain corresponds to multiplication by the variable in the other.
Multiplication by an exponential in one domain corresponds to a shift (or delay) in the other.

Transformation of Derivatives:

\[
L \left( \frac{df(t)}{dt} \right) = \int_0^\infty e^{-st} \frac{df(t)}{dt} \, dt = \left[ e^{-st} f(t) \right]_0^\infty - \int_0^\infty -se^{st} f(t) \, dt
\]

Initial value of \( f(t) \) at \( t=0 \).

\[
= -f(0) + sF(s).
\]

Following same pattern,

\[
L \left( \frac{d^2f(t)}{dt^2} \right) = \int_0^\infty e^{-st} \frac{d^2f(t)}{dt^2} \, dt = \left[ e^{-st} \frac{df(t)}{dt} \right]_0^\infty - \int_0^\infty se^{-st} f(t) \, dt
\]

\[= -f(0) - sf(0) + s^2F(s).\]

Transformation of Ordinary Diff. eqn.

\[
M \frac{d^2x(t)}{dt^2} + c \frac{dx(t)}{dt} + kx(t) = f(t)
\]

\[
L \{ x(t) \} = X(s)
\]

\[
L \{ f(t) \} = F(s)
\]

\[
M \left[ s^2 X(s) - sx(0) - \dot{x}(0) \right] + c \left[ sX(s) - x(0) \right] + kX(s) = F(s)
\]

Initial displacements and velocities respectively.
\[ \frac{\xi}{m} = 2 \frac{\xi}{\nu} \]

\[ \frac{k}{m} = \omega_n^2 \]

\[ X(s) = \frac{1}{m} \frac{F(s)}{s^2 + 2 \xi \omega_n s + \omega_n^2} \]

**Shifting Theorem**

A frequency encountered in \( f(t) = f(t) e^{at} \)

\[ L \left[ f(t) e^{at} \right] = F(s-a) \]

Effect of multiplying \( f(t) \) by \( e^{at} \) in time domain is to shift the \( L \)-place Transform \( F(s) \) by \( a \) in the \( s \)-domain.

**Determine the response of a Mass-Spring System to the One-Sided Harmonic Excitation?**

\[ F(t) = F_0 \sin \omega t u(t) \]

**Shock Spectrum**

Structures are subjected to large forces applied suddenly and over periods of time that are short relative to the natural period of the structure. → Undesirable vibration → Large cyclic stress → Diminishing the structure and performance.

A form of this type → Shock.

The severity of the shock is generally measured in terms of the Maximum value of the response.
A shock \( F(t) \) is characterized by its maximum value \( F_0 \), its duration \( T \), its shape or alternatively, the impulse

\[
\int_0^T F(t) \, dt.
\]

Assume the reasonable approximation for the force in the Half-Sine pulse.

**Half-Sine pulse → Superposition of two One-Sided Sine fins**

1. One initiated at \( t = 0 \)
2. (Instantiated at \( t = T = T/\omega \))

\[
F(t) = F_0 \left[ u(t) - u(t-T) \right]
\]

\[
= F_0 \left[ \sin(\omega t) + \sin(\omega t) \right]
\]

Usual procedure to describe the half-Sine pulse.

The response to half-Sine pulse is

\[
x(t) = \frac{F_0}{k} \frac{1}{1 - (\omega/\omega_n)^2} \left\{ \left( \sin(\omega t) - \frac{\omega}{\omega_n} \cos(\omega t) \right) u(t) + \left[ \sin(\omega(t-T)) - \frac{\omega}{\omega_n} \cos(\omega(t-T)) \right] u(t-T) \right\}
\]

The maximum response,

\[
x(t) = \frac{F_0}{k} \frac{1}{1 - (\omega/\omega_n)^2} \left[ \sin(\omega t) - \frac{\omega}{\omega_n} \cos(\omega t) \right], \quad 0 < t < T/\omega
\]

we take this component.
To get maximum response, find \( \frac{dx(t)}{dt} = 0 \), and get the time \( t_m \) at which \( \frac{dx}{dt} = 0 \). Then substitute \( t_m \) in above eqn (2).

differentiate, get:
\[
x(t) = \frac{F_0 \omega}{k \left[ 1 - \left( \frac{\omega}{\omega_n} \right)^2 \right]} \left( \cos \omega t - \cos \omega_n t \right)
\]

Using the relation, \( \cos \alpha - \cos \beta = -2 \sin \frac{\alpha + \beta}{2} \sin \frac{\alpha - \beta}{2} \).

\[
\sin \left( \frac{\omega_n t + \omega t}{2} \right) \sin \left( \frac{\omega_n t - \omega t}{2} \right) t_m = 0
\]

which has two families of solutions:
\[
t_m = \frac{\omega n \pi}{2} i = 1, 2, \ldots
\]

Substitute for \( t_m \) in (2) we get,
\[
x(t_m) = \frac{F_0}{k} \frac{\sin \frac{\omega_n \pi t_m}{\omega}}{1 + \omega_n \omega}
\]
\[
x(t_m) = \frac{F_0}{k} \frac{\sin \frac{\omega n \pi t_m}{\omega}}{1 - \omega_n \omega}
\]

The response corresponding to \( t = t_m \) achieves higher values than the other. The response corresponding to \( t = t_m \).

The value of integer \( i \) to be determined,

\( t_m \) must occur during the pulse.

\[
\left| \frac{\omega n \pi}{(\omega_n + \omega)} \right| < \frac{\pi}{\omega}
\]

Hence we conclude that for \( 0 < t < \frac{\pi}{\omega} \) we have the max. response.

\[
x_{\text{max}} = \frac{F_0 \omega n}{w} \frac{\omega n \pi}{k \left( \frac{\omega_n}{\omega} - 1 \right)} \sin \frac{\omega n \pi}{1 + \omega_n \omega}, \quad i < \frac{1}{2} \left( 1 + \frac{\omega_n}{\omega} \right).
\]
The response for any time after the termination of the pulse can be verified to be,

\[ x(t) = \frac{F_0 \omega_n^2}{\omega} \frac{\cos (\omega_n t)}{k \left[ 1 - \left( \frac{\omega_n}{\omega} \right)^2 \right]} \left[ \sin \omega_n T + \sin \omega_n (t-T) \right], \quad t > T_0 \]

For \( B \) obtains the max. response, we must 1st determine, \( t = t_m \)

which time \( \dot{x}(t) = 0 \)

\[ \dot{x}(t) = \frac{F_0 \omega_n^2}{\omega} \left[ \cos (\omega_n t) + \cos \omega_n (t-T) \right] \]

\[ \cos x + \cos y = 2 \cos \left( \frac{x+y}{2} \right) \cos \left( \frac{x-y}{2} \right) \]

we conclude that \( t_m \) must satisfy the eqn,

\[ \cos \omega_n \left( t_m - \frac{1}{2} T \right) \cos \left( \frac{1}{2} \omega_n T \right) = 0 \]

which yields the soln,

\[ t_m = \left( \omega_n^2 - 1 \right) \frac{1}{\omega_n} + \frac{T}{2}, \quad i = 1, 2, \ldots \]

Substitute \( t_m \) in \( x(t) \).

\[ x(t) = \frac{F_0 \omega_n}{k} \max \left[ 1 - \left( \frac{\omega_n}{\omega} \right)^2 \right] \]

\[ x_{\max} = \frac{F_0}{k} \]

\[ \omega_n \frac{\pi}{\omega} \]

\[ a^{-1} \]

\[ \frac{1}{\omega_n} \]

\[ b^{-1} \]

\[ \frac{1}{\omega_n} \]

\[ t > \frac{\pi}{\omega} \]

\[ k \left[ 1 - \left( \frac{\omega_n}{\omega} \right)^2 \right] \]
The plot is $X_{\text{max}} = w/\omega$ in which $X_{\text{max}}$ is the maximum value of $X$ in a pulse with $i = \frac{1}{2}(1 + \omega^2)$ and

$X_{\text{max}}$ without $i$ are considered.

\[ i \text{ gives } X_{\text{max}} \text{ for } \omega < w \]

\[ \omega \text{ is not valid for } \omega < w \]

\[ \omega \text{ is valid for } \omega > w \]

In graph, $\frac{X_{\text{max}}}{F_0}$ is non-dimensional.

For different pulse shapes, different shock spectra.

For regular, regular pulses, $\frac{w}{\omega}$ has no meaning, being no defined in the pulses.

But $T$ is defined, so $T/T_0$ can be plotted with $X_{\text{max}}/F_0$

System Response by the Laplace Transformation Method: — Transfer function

The solution to many problems in vibrations by direct means.

— Some difficulty. — so transformation is used — There is a very large variety of transformations. — general idea behind all of them is the same.

— Transform a difficult problem to simple one.

— Solve the simple problem $x = t$.

— Inverse transform the solution of the simple problem.

— Here we use Laplace Transformation.

One-sided L-T of $x(t)$ is $X(s) = \mathcal{L} \{x(t)\}$.

\[ X(s) = \mathcal{L} \{x(t)\} = \int_{0}^{\infty} e^{-st} x(t) \, dt \]
\( e^{-st} \) kernel of transformation

\[ \int_{-\infty}^{\infty} e^{-st} f(t) dt \]

\[ L \{ f(t) \} = \int_{0}^{\infty} e^{-st} f(t) dt \]

It is necessary to evaluate the transforms of the derivatives.

\[ \frac{dx}{dt}, \quad \frac{d^2x}{dt^2} \]

\[ L \left( \frac{dx}{dt} \right) = \int_{0}^{\infty} e^{-st} \frac{dx}{dt} dt = e^{-st} x(t) \bigg|_{0}^{\infty} + \int_{0}^{\infty} e^{-st} \frac{d^2x}{dt^2} dt \]

\[ = s^2 X(s) - s x(0) \]

where \( x(0) \) value of fn \( x(t) \) at \( t = 0 \).

It represents the displacement of the mass \( m' \).

Similarly, it is not difficult to show that:

\[ L \left( \frac{d^2x}{dt^2} \right) = \int_{0}^{\infty} e^{-st} \frac{d^2x}{dt^2} dt = \int_{0}^{\infty} e^{-st} \frac{dx}{dt} dt \]

\[ = -s x(0) + \int_{0}^{\infty} e^{-st} \frac{d^2x}{dt^2} dt \]

\[ = s^2 X(s) - s x(0) - x'(0) \]

Laplace transformation of the equation for \( x(t) \) is simply,

\[ F(s) = L \{ F(t) \} = \int_{0}^{\infty} e^{-st} F(t) dt \]

Transforming and rearranging:

\[ (ms^2 + cs + k) X(s) = F(s) + mx(0) + (ms + c)x(0). \]
\((m^2 + cs + k)x(s) \rightarrow \text{a general transformed equation}\)

So solution to homogeneous in \(s\).

Let \(x(0) = 0\)

So the transformed equation

\[ Z(s) = F(s) = m^2 + cs + k \]

Transformed response.

\[ G(s) = \frac{1}{Z(s)} = \frac{X(s)}{F(s)} = \frac{1}{m^2 + cs + k}. \]

Impedance Transfer.

Called System fn or transfer fn.

\[ G(j\omega) = \frac{1}{m^2 + j2\pi\omega + \omega^2}. \]

Represents an algebraic expression in \(s\) plane.

Generally a complex plane called Laplace plane or Laplace domain.

Now \(s = j\omega\) in \(G(s)\) and multiply by \(m\).

\[ G(j\omega) = \frac{\theta}{m} \cdot \epsilon \times j\omega. \]

Frequency response.

\[ B(j\omega) = \frac{1}{1 - \left(\frac{j\omega}{\omega_n}\right)^2 + j\omega \frac{\theta}{\omega_n}}. \]
So eqn, \[ X(s) = G(s) F(s) \]

**Transform response = Transfer fn**

**Final step = Inverse transform**
- Use Inverse Laplace transform

\[ x(t) = L^{-1} X(s) = L^{-1} G(s) F(s) \]

- \( F(s) \) is a line integral in the complex s-domain
- Method of partial fractions
- \( R(s) \)
  - L-circuit step fn
  - Ramp fn
  - Delta fn

**Ex:** SDOF, undamped

\[ m \ddot{x} + c \dot{x} + kx = 0 \]

Plot response for \( T_0 = 0.4 \)

\[ \omega_n = 4 \text{ rad/s} \]

\[ C = 0, \quad (\omega_n^2 + \omega_n^2) \to F(t) \to C \]

\[ F(t) = \frac{F_0}{T_0} 0 < \frac{t}{T_0} \]

\[ x(t) = L^{-1} G(s) F(s) \]

\[ G(s) \text{ from } \Omega \to \frac{1}{ms^2 + k} = \frac{1}{m(s^2 + \omega_n^2)} \]
\[ f(t) = \int_0^\infty e^{-st} f(t') dt' = F_0 \int_0^T e^{-st} dt = F_0 \left( \frac{e^{-sT_0}}{-s} + \frac{sT_0}{e^{-sT_0}} \right) \]

\[ = \frac{F_0}{T_0} \left( -\frac{e^{-sT_0}}{s} - T_0 + e^{-sT_0} T_0 \right) \]

\[ = \frac{F_0}{T_0} \left[ \frac{T_0 - T_0^2}{s} + \frac{T_0^2}{e^{sT_0}} \right] = \frac{F_0}{T_0} \left[ \frac{T_0^2}{s} + \frac{T_0^2}{1 - e^{-sT_0}} \right] \]

\[ L^{-1} \left[ \frac{1}{s(s^2 + w_n^2)} \right] = \frac{1}{w_n^2} \left( 1 - \cos w_n t \right) \]

\[ L^{-1} \left[ \frac{1}{s^2(s^2 + w_n^2)} \right] = \frac{1}{w_n^3} \left( w_n t - \sin w_n t \right) \]

\[ n(t) = \frac{F_0}{mT_0} \left\{ -\frac{T_0}{w_n^2} \left[ 1 - \cos w_n (t - T_{0}) \right] u(t - T_{0}) + \frac{1}{w_n^3} \left( w_n t - \sin w_n t \right) u(t - T_{0}) \right\} \]

\[ \frac{1}{2} = \frac{1}{w_n^2} \left[ w_n (t - T_{0}) - \sin w_n (t - T_{0}) u(t - T_{0}) \right] \]

\[ \frac{1}{2} = \frac{1}{w_n^2} \left[ w_n (t - T_{0}) - \sin w_n (t - T_{0}) u(t - T_{0}) \right] \]

\[ \text{Unknown fields}\]
Impulse Response is equal to the Inverse $L^{-1}$ of the Transfer fn.

Delta fn $L^{-1}$ of Unit Impulse is:

$$A(s) = \int_0^\infty e^{-st} g(t) \, dt = \frac{-st}{-st} \left| _{t=0}^{\infty} \right. = \frac{1}{s} - \frac{1}{s}$$

$$x(t) = g(t)$$

$$F(s) = A(s) = 1 \left( \text{unit step} - \text{unit impulse} \right) = \frac{1}{s} - \frac{1}{s^2}$$

$$e_r \quad g(t) = L^{-1} G(s) \cdot A(s) = \frac{1}{s} L^{-1} G(s)$$

Step fn $L^{-1}$ of step fn:

$$a(s) = \int_0^\infty e^{-st} u(t) \, dt = \frac{1}{s^2}$$

$$x(t) = a(t)$$

$$F(s) = U(s) = \frac{1}{s}.$$ 

$$a(t) = L^{-1} G(s) \cdot F(s) = L^{-1} g(s) \cdot U(s) = L^{-1} G(s) \cdot \frac{1}{s}$$

$$= \frac{1}{s} \frac{L^{-1} G(s)}{s} \quad \text{Laplace transform of transfer fn divided by } s^3.$$

Ramp fn $L^{-1}$ of Ramp fn is:

$$R(s) = \int_0^\infty e^{-st} x(t) \, dt$$

$$= \int_0^\infty e^{-st} t \cdot u(t) \, dt = \int_0^\infty e^{-st} t \, dt = \frac{-e^{-st}}{-s} \bigg|_0^\infty$$

$$\frac{1}{s} \int_0^\infty e^{-st} \, dt = \frac{1}{s^2}$$

$$x(t) = L^{-1} G(s) \cdot F(s) = L^{-1} G(s) \cdot \frac{1}{s}$$

$$a(t) = L^{-1} g(s) \cdot F(s) = L^{-1} g(s) \cdot \frac{1}{s}$$
General System Response

\[ F(t) \rightarrow \text{arbitrary excitation} \]

\[
\begin{align*}
    x(0) &= x_0 \\
    x'(0) &= v_0
\end{align*}
\]

Transform both sides of, \( m \ddot{x}(t) + c \dot{x}(t) + kx(t) = F(t) \).

\[ \bullet \quad x(s) = \mathcal{L}[x(t)] = \int_0^\infty e^{-st} x(t) dt \]

\[ \dot{x}(s) = \mathcal{L}[\dot{x}(t)] = \int_0^\infty e^{-st} \frac{d}{dt} x(t) dt = -s \cdot x(s) \quad x(0) \text{ is the } x(t) \text{ at } t = 0 \]

\[ \mathcal{L}[x(t)] \mathcal{L}[\dot{x}(t)] \mathcal{L}[\ddot{x}(t)] = \int_0^\infty x(t) \cdot e^{-st} dt = s^2 \cdot x(s) - s \cdot x(0) - x'(0) \]

\[
M \left[ s^2 x(s) - s x(0) - x'(0) \right] + C \left[ s x(s) - x(0) \right] + k x(s) = F(s)
\]

\[ (ms^2 + cs + k) x(s) = (ms - c) x(0) - m \ddot{x}(0) = F(s) \]

For a spring-mass-damper system, \( x(0) = 0 \), \( x'(0) = 0 \).

\[ u_1 (ms^2 + cs + k) x(s) = F(s) \]

\[ u_1 = \text{an algebraic eqn.} \]

\[
\begin{align*}
    T_0 \left[ T_0 (J - \omega^2) \right] \ddot{x}(s) &= F(s) \\
    m \left[ \frac{s^2 + 2 \omega \omega_n s + \omega_n}{s^2 + 2 \omega \omega_n s + \omega_n^2} \right] &= \frac{\omega_n}{\omega} \cdot \frac{m}{10}
\end{align*}
\]
\[ x(t) = x_0 + \int_0^t f(\tau) d\tau \]

\[ x(t) = x_0 + \int_0^t f(\tau) d\tau \]

**For Inverse L^T, consider each term separately.**

**1st term, \( \frac{f(s)}{m(s^2 + 2\delta_m s + \omega_m^2)} \)**

The inverse L^T of the product of two transforms is equal to the convolution of their inverse transforms.

Let \( f_1(t) \) and \( f_2(t) \) be two fns\'s defined for \( t > 0 \).

\( f_1(t) \) and \( f_2(t) \) possess L^T, \( F_1(s) \) and \( F_2(s) \) respectively.

Consider the integral:

\[ x(t) = \int_0^t f_1(\tau) f_2(t-\tau) d\tau \]

The fn \( X(t) \), denoted by \( x(t) = \int_0^t f_1(\tau) f_2(t-\tau) d\tau \) is called convolution of fns \( f_1 \) & \( f_2 \) over \( 0 < t < \infty \).

The upper limits are inter-changeable, e.g., \( f_1(\tau) f_2(t-\tau) = 0 \) for \( \tau > t \).

which is same as \( t - \tau < 0 \).

**Transform both sides, we get,**

\[ X(s) = \int_0^\infty e^{-st} \left[ \int_0^\infty f_1(\tau) f_2(t-\tau) d\tau \right] dt \]

\[ = \int_0^\infty f_1(\tau) \left[ \int_0^\infty e^{-st} f_2(\tau) \right] d\tau dt \]

\[ = \int_0^\infty f_1(\tau) \left[ \int_0^\infty e^{-st} f_2(t-\tau) d\tau \right] dt \]
Lower limit of 2nd integral changed without affecting the result by \( f_2(t-T) = 0 \) for \( t < T \).

The transformation \( t - T = \lambda \) in the last integral:

\[
X(s) = \int_0^\infty f_1(\lambda) \cdot \left[ \int_0^\infty e^{-s(\lambda + \lambda)} f_2(\lambda) \, d\lambda \right] \, d\lambda
\]

\[
= \int_0^\infty e^{-s\lambda} f_1(\lambda) \, d\lambda \cdot \int_0^\infty e^{-s\lambda} f_2(\lambda) \, d\lambda = F_1(s) \cdot F_2(s).
\]

\[
X(t) = L^{-1}X(s) = L^{-1}F_1(s) \cdot F_2(s)
\]

\[
F_1(s) = F(s) \quad \text{and} \quad \int_0^t f_1(\tau) \, d\tau = \int_0^t f_1(\tau) \, d\tau
\]

\[
f_2(t) = g(t) \rightarrow \text{Impulse response}.
\]

Inverse L.T of 1st term is \( L^{-1}F_1(s)F_2(s) = \int_0^t f_1(\tau) \cdot f_2(t-\tau) \, d\tau \)

\[
= \frac{1}{Mw_d} \int_0^t F(\tau) e^{-\frac{\sqrt{2} \omega_n}{w_d} (t-\tau)} \sin \omega_d(t-\tau) \, d\tau
\]

For 2nd term, inverse L.T is,

\[
L^{-1} \frac{S + 2\sqrt{2} \omega_n}{S^2 + 2\sqrt{2} \omega_n s + \omega_n^2} = \frac{\omega_n e^{-\frac{\sqrt{2} \omega_n}{w_d} \cos (\omega_d t - \psi)}}{w_d}
\]

For 3rd term, inverse L.T is, \( u_j = f_2(t) \cdot m \).
So the general response,

\[ x(t) = \frac{1}{m \omega_d} \int_0^t F(t') e^{-\omega_{nt} t} \sin \omega_d (t-t') \, dt' \]

\[ + \frac{x_0 \omega_n}{\omega_d} e^{-\omega_{nt}} \cos (\omega_d t - \psi) + \frac{v_0}{\omega_d} e^{-\omega_{nt}} \sin \omega_d t \]

L-T method permits to produce both the response to the initial conditions and the response to the external excitation.

\[ L^{-1} F_1(s) F_2(s) = \int_0^t f_1(t-t') f_2(t') \, dt' \]

\[ = \frac{1}{m \omega_d} \int_0^t F(t-t') e^{-\omega_{nt}} \sin \omega_d t \, dt' \]

So the general response have alternative form,

\[ x(t) = \frac{1}{m \omega_d} \int_0^t F(t-t') e^{-\omega_{nt}} \sin \omega_d t \, dt' \]

\[ + \frac{x_0 \omega_n}{\omega_d} e^{-\omega_{nt}} \cos (\omega_d t - \psi) + \frac{v_0}{\omega_d} e^{-\omega_{nt}} \sin \omega_d t \]

\[ \int_{t=0}^{t=t'} F(t-t') \, dt' \]

\[ \text{Where} \]

\[ \omega_d = \sqrt{\frac{k}{m}} \]

\[ \omega_n = \sqrt{\frac{k}{m}} \]

\[ T = \frac{\pi}{\omega_d} \]

\[ \eta = \frac{1}{\omega_d} \]

\[ \xi = \frac{1}{2} \]
Response By the State Transition Matrix

Consider,

\[ x(t) = \frac{1}{m\omega_d} \int_0^t F(T) e^{-\frac{\xi}{\omega_d} (t-T)} \sin \omega_d (t-T) dT \]

\[ + x_0 e^{-\frac{\xi}{\omega_d} t} \cos (\omega_d t - \phi) + \frac{v_0}{\omega_d} e^{-\frac{\xi}{\omega_d} t} \sin \omega_d t \]

as the system response to Damped SDOF to any arbitrary excitation \( F(t) \)

in terms of convolution integral.

Finding Solution to response.

**Numerical Solution.**

2nd Order D.E Can be transformed to 1st Order D.E.

We write the eqn of motion of a Damped SDOF system,

\[ mI(t) + 2\xi x(t) + \omega_n^2 x(t) = mF(t) \]

Introduce,

\[ x(t) = x_1(t) \]

\[ \dot{x}(t) = x_2(t) \]

\[ \dot{x}_1(t) = x(t) = x_2(t) \]

\[ \dot{x}_2(t) = \ddot{x}(t) = -\omega_n^2 x(t) - 2\xi x(t) + mF(t) \]

\[ = -\omega_n^2 x_1(t) - 2\xi x_2(t) + mF(t) \]

The pair of variables \( x_1(t) \)

\( x_2(t) \) or \( \dot{x}(t) \) or \( \xi(t) \)

For any set of initial conditions \( x(0) = x_1(0) \)

\( \dot{x}(0) = x_2(0) \)

For any future time,

The state eqns define the state of the system.

The soln to (i) can be best presented in terms of matrix

Notations: \( \dot{x} \) = Called State vector.
\[ x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \]

Consequently, \( x(t) \) is multiplied by a 2x2 matrix \( \mathbf{K}(t) \).

Then get a solution for \( \mathbf{k}(t) \).

Then for \( \mathbf{K}(t) \), multiply by \( \mathbf{k}^{-1}(t) \) to get \( x(t) \).

\[ x(t) = \phi(t) x(0) + \int_0^t \phi(t-\tau) b f(\tau) d\tau \]

\( x(0) = \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} \) as the initial state vector.

\[ \phi(t-\tau) = e^{\mathbf{A}(t-\tau)} = \mathbf{I} + (t-\tau) \mathbf{A} + \frac{(t-\tau)^2}{2!} \mathbf{A}^2 + \frac{(t-\tau)^3}{3!} \mathbf{A}^3 + \ldots \]

is known as the state transition matrix.